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The Asymptotic Expansion of Integral Functions Defined by Taylor's Series

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V. *The Asymptotic Expansion of Integral Functions defined by* Taylor's *Series.*

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Contents.

INTRODUCTION.

§ 1. Integral functions can be defined either by TAYLOR'S series or Weierstrassian products. When the zeros are simple functions of their order number, the latter method is, as a rule, most simple. When the zeros, however, are transcendental functions of the order number, those integral functions which so far have occurred in analysis have been defined by TAYLOR'S series.

[Definitions by definite integrals have usually been reducible to one of the preceding forms.]

Whatever be the manner of its definition, an integral function has a single essential singularity at infinity, and the behaviour near this singularity serves to classify the function. By studying this behaviour we may hope to find connecting links between the two modes of definition.

The behaviour at infinity is determined by asymptotic expansions.

The first expansion of a function was derived from STIRLING'S[1] approximation to $n\,!$. This led naturally to expressions for $\Gamma(x)$ when x is large and real.

[1] STIRLING, 'Methodus Differentialis,' 1730.

Such were considered by, among others, CAUCHY,[1] BINET,[2] and RAABE.[3] Other references to the history of the subject will be found in the "Encyklopädie der Mathematischen Wissenschaften."[4]

But the behaviour of a function defined by a Weierstrassian product, when considered only for real values of the variable near infinity, affords little knowledge of the essential singularity. STIELTJES[5] first proved the asymptotic expansion for $\Gamma(x)$ to be valid for all values of $|\arg x| < \pi$. His result was subsequently obtained by MELLIN.[6] Immediately afterwards the author,[7] from an idea suggested by one of MELLIN'S earlier papers and due originally to RIEMANN,[8] extended the result to the multiple gamma functions. Then, simultaneously, MELLIN[9] and the author[10] discovered the asymptotic expansions for large classes of integral functions defined by Weierstrassian products. Such investigations have been developed by the author in a series of papers.[11]

It is natural to expect that similar results can be obtained for functions defined by TAYLOR'S series.

An asymptotic expansion for BESSEL'S function $J_0(x)$ was first given for real values of x by POISSON.[12] The result was extended to other integral values of n, that is to say, to functions $J_n(x)$, where n is an integer, by JACOBI.[13] Then, in a noteworthy paper, HANKEL[14] extended the result to general complex values both of the parameter n and the variable x; and though his statement of his results merited the criticism of HURWITZ,[15] it deserves recognition as a valuable discovery. The question has since been considered, among others, by WEBER[16] and NIELSEN[17]. Further references will be found in the 'Encyklopädie'[18] and in NIELSEN'S text book.[17]

In this connection mention may be made of a similar investigation by HOBSON[19] in the theory of LEGENDRE'S functions.

Closely allied to BESSEL'S function are integral functions defined by generalised hypergeometric

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[1] CAUCHY, 'Exercices d'Analyse,' tome 2, p. 386.

[2] BINET, 'Journal de l'École Polytechnique,' tome 27, p. 220.

[3] RAABE, 'Crelle,' vol. 25, p. 147; vol. 28, p. 10.

[4] BRUNEL, *loc. cit.*, vol. 2, A, p. 166.

[5] STIELTJES, 'Liouville,' sér. 4, vol. 5, p. 425.

[6] MELLIN, 'Acta Societatis Scientiarum Fennicae,' tome 24, No. 10.

[7] BARNES, 'Phil. Trans. Roy. Soc.,' A, vol. 196, p. 265.

[8] RIEMANN, 'Œuvres,' 1898, p. 165.

[9] MELLIN, 'Acta Societatis Scientiarum Fennicae,' tome 29, No. 4.

[10] BARNES, 'Phil. Trans. Roy. Soc.,' A, vol. 199, pp. 411–500.

[11] BARNES, 'Cambridge Phil. Trans.,' vol. 19, pp. 322–355; pp. 426–429; 'Proc. Lond. Math. Soc.,' ser. 2, vol. 3, pp. 253–272, and pp. 273–295.

[12] POISSON, 'Journal de l'École Polytechnique,' tome 19, p. 349.

[13] JACOBI, 'Gesammelte Werke,' vol. 7, p. 174.

[14] HANKEL, 'Mathematische Annalen,' vol. 1, pp. 467–501.

[One of the referees has pointed out that I had omitted to mention the brilliant investigation of STOKES, which remained long unknown to continental mathematicians. STOKES obtained the asymptotic expansions of the solutions of BESSEL'S equation for complex values of the variable in two papers published in 1857 and 1868 respectively ('Cambridge Philosophical Transactions,' vol. 10, p. 105; vol. 11, p. 412). The reader may also notice STOKES' 'Cambridge Philosophical Proceedings,' vol. 6, p. 362, and 'Acta Mathematica,' vol. 26, pp. 393–397.]

[15] HURWITZ, 'Mathematische Annalen,' vol. 33, p. 246.

[16] WEBER, 'Mathematische Annalen,' vol. 6, p. 148.

[17] NIELSEN, 'Handbuch der Cylinderfunctionen,' 1904, pp. 156, &c.

[18] WANGERIN, 'Encyklopädie der Mathematischen Wissenschaften,' Band 2, A, p. 748.

[19] HOBSON, 'Phil. Trans.,' A, vol. 187, pp. 443–531.

functions. Here for real values of the variable STOKES[1] first gave asymptotic expansions, and ORR[2] has recently extended his results to general complex values of the argument.

Quite recently MITTAG-LEFFLER[3] has constructed the new function $E_a(x)$ and investigated its asymptotic behaviour.

It was, however, in the theory of linear differential equations that POINCARÉ[4] first pointed out the use of divergent series as solutions in the neighbourhood of infinity, and laid the foundation of a rigorous theory of such series. The continuation of his investigations has been the subject of many researches, notably by KNESER and HORN. For references in this connection I may refer the reader to FORSYTH'S 'Theory of Differential Equations.'[5]

Another connected series of investigations may be mentioned. HADAMARD[6] first gave a remarkable theorem as to the maximum value of the modulus of an integral function defined by TAYLOR'S series on a circle of large radius. Other theorems of similar type are due to BOREL[7] and BOUTROUX.[8] Valuable, however, as such theorems are on account of their generality, we need complete asymptotic expansions before we can adequately classify integral functions. Further references will be found in BOREL'S[9] text-book.

There is a close connection between the asymptotic expansions of certain types of integral functions and what BOREL[10] has called the associated functions defined by TAYLOR'S series of finite radius of convergence. This connection enables us to investigate the singularities of many types of such TAYLOR'S series, and thus connects the theory with a whole series of investigations. Reference may be made to the work of FABRY,[11] LE ROY,[12] LINDELÖF,[13] and LEAU.[14] A very complete bibliography of this branch of modern mathematics is given by HADAMARD.[15]

§ 2. In the present paper the author attempts to give unity to the investigations of asymptotic expansions of integral functions defined by TAYLOR'S series by taking various standard types of such functions and applying new methods of contour integration so as to get, as simply and elegantly as possible, complete asymptotic expansions. For each function investigated we find the nature of the behaviour at infinity. The investigation may be regarded as preliminary to the formation of a classified table: it is complementary to that previously carried out for functions defined as products.

It is hardly necessary, perhaps, to say that no methods, however powerful, will apply to every function that can be constructed by a TAYLOR'S series. Just as, in general, a TAYLOR'S series admits its circle of convergence as a line of essential singularity, so the general integral function, which we may define by a TAYLOR'S series, will not admit the same dominant asymptotic expansion for any range of values of arg x, however small.

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1 STOKES, 'Cambridge Phil. Soc. Proc.,' vol. 6, pp. 362–366.

2 ORR, 'Cambridge Phil. Soc. Trans.,' vol. 17, pp. 171–200; pp. 283–290.

3 MITTAG-LEFFLER, 'Comptes Rendus,' vol. 137, pp. 554–558; 'Acta Mathematica,' vol. 29.

4 POINCARÉ, 'Acta Mathematica,' tome 8, pp. 295–344.

5 FORSYTH, *loc. cit.*, Part III., vol. 4, 1902, p. 341.

6 HADAMARD, 'Liouville,' sér. 4, vol. 9, pp. 171–215.

7 BOREL, 'Acta Mathematica,' vol. 20, pp. 357–396.

8 BOUTROUX, 'Acta Mathematica,' vol. 28, pp. 1–128.

9 BOREL, 'Leçons sur les Fonctions Entières,' 1900.

10 In the memoir just cited.

11 FABRY, 'Annales de l'École Normale Supérieure,' sér. 3, tome 13, pp. 367–399; 'Acta Mathematica,' tome 22, pp. 65–87; 'Liouville,' sér. 5, tome 4, pp. 317–358.

12 LE ROY, 'Annales de la Faculté des Sciences de Toulouse,' sér. 2, tome 2 (1900).

13 LINDELÖF, 'Acta Societatis Scientiarum Fennicae,' tome 24, No. 7.

14 LEAU, 'Liouville,' sér. 5, tome 5, pp. 365–425.

15 HADAMARD, "La Série de Taylor et son prolongement analytique" ('Scientia,' 1901).

In conclusion, I must mention a paper of HARDY[1] in which he obtains some of the present results. He was led to the question of asymptotic behaviour by a desire to obtain approximations for the large zeros of integral functions, one of the subsidiary problems which a general knowledge of integral functions will solve. His paper was sent to me in August, 1904, in the capacity of referee to the London Mathematical Society. He had obtained the first terms of the asymptotic forms of the function which I call $G_\beta(x; \theta)$ in the case where β and θ are real. In my reply I said that I had already obtained complete expansions for general complex values of β and θ. Such results Mr. HARDY has since published in the revised form of his paper. The reader will find it instructive to compare our respective discussions of the question.

[*Note added March* 21, 1906.—The Council of the Royal Society suggested that the paper in its original form contained so many developments that it was more of the nature of a treatise than of a paper to be published in their Transactions. In consequence it has been considerably compressed, and statements of results have been in many cases given in lieu of detailed investigations. Developments of such a nature will, I hope, with my subsidiary investigations, be suitable for publication elsewhere. In compressing the paper, certain changes have at times been made in the mode of presentation. Whenever such a change has been made, or whenever a result has been stated which had not been originally obtained, the number of the corresponding paragraph is placed in square brackets [ ].]

Preliminary Definitions and Theorems.

§ 3. The function $f(x)$ is said to admit the asymptotic expansion

$$c_0 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots$$

for a given range of values of arg x, when $|x|$ is large, if the following condition is satisfied. We put

$$\left| f(x) - c_0 - \frac{c_1}{x} - \dots - \frac{c_n}{x^n} \right| = R_n.$$

Then it must be possible, for any assigned value of n, to find a value X for $|x|$ such that, whenever

$$|x| > X, \quad |x^n R_n| < \epsilon,$$

where ϵ is any arbitrarily assigned small positive quantity.

The solution of linear differential equations often gives rise to series of which the simplest type is

$$e^x \left[c_0 + \frac{c_1}{x} + \dots + \frac{c_n}{x^n} + \dots \right],$$

the series within the brackets being divergent.

We say that $f(x)$ is asymptotically represented by such a series for any value or range of values of arg x under the following conditions.

Put

$$\left| f(x) - e^x \left[c_0 + \frac{c_1}{x} + \dots + \frac{c_n}{x^n} \right] \right| = R_n.$$

[1] HARDY, "On the Zeros of Certain Classes of TAYLOR'S Series," Part II., 'Proc. Lond Math. Soc.,' ser. 2, vol 2, pp. 401–431.

Then for any assigned value of n it is possible to find a value X for $|x|$ such that, whenever $|x| > X$,

$$|e^{-x}x^n R_n| < \epsilon,$$

where ϵ is an arbitrarily assigned positive quantity.

It is evidently possible that an asymptotic expansion may hold for some values of $\arg x$ and not for others.

§ 4. The following definitions give precision to subsequent statements.

When we say of a quantity $J(x, k)$ that, for any assigned value or range of values of $\arg x$, it is of order less than $1/|x|^k$ when $|x|$ is large, we mean that for any assigned value of k it is possible to find a value X of x such that, when $|x| > X$,

$$|J(x, k)\, x^k| < \epsilon,$$

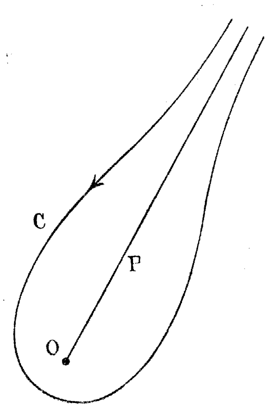
ϵ being defined as before.

When we say that $J(x, k)$ tends exponentially to zero with $1/|x|$, we mean that it is such that, when $|x| > X$ and $\mathfrak{R}(x) > 0$,

$$|J(x, k)\, e^{px}| < \epsilon,$$

p being a definite finite positive quantity.

§ 5. Our fundamental procedure is based upon the following theorem.



Suppose that, when $|x|$ is large, we wish to find an asymptotic expansion for the integral

$$I = \frac{\iota}{2\pi}\int_C e^{-xz} f(z)\,(-z)^{\beta-1}\,dz.$$

The integral is taken round a gamma-function contour C which encloses the origin and embraces an axis P from the origin to infinity along which $\mathfrak{R}(xz)$ is positive.

In the subject of integration $f(z)$ is a function which, for values of $|z| < l$, admits the convergent expansion

$$f(z) = \sum_{n=0}^{\infty} c_n (-z)^n.$$

Further, $f(z)$ is such that the integral I is convergent. This condition, of course, limits the behaviour of $f(z)$ at infinity along the axis P. Suppose that the plane of the complex variable z is dissected by lines passing away from the poles of $f(z)$ to infinity in a direction away from the origin. We assume that the contour C does not contain or cut any of these lines.

Then the integral I admits the asymptotic expansion

$$\sum_{n=0}^{\infty} c_n \frac{\iota}{2\pi}\int_C e^{-xz}(-z)^{\beta+n-1}\,dz = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(1-\beta-n)\, x^{\beta+n}}.$$

Divide up the contour C into two parts L and M. L lies wholly within the circle of convergence of $f(z)$ and, on L, $|z| \gtrless l'$, where $l' = l - \epsilon$ and ϵ is a positive quantity as small as we please. M forms the remainder of the contour.

We have

$$\mathrm{I} - \sum_{n=0}^{k-1} \frac{c_n}{\Gamma(1-\beta-n)\,x^{\beta+n}} = \frac{\iota}{2\pi}\int_{\mathrm{C}} e^{-xz}\left\{f(z) - \sum_{n=0}^{k-1} c_n(-z)^n\right\}(-z)^{\beta-1}\,dz$$

$= \mathrm{I}_1 + \mathrm{I}_2$ (say), where I_1 is the integral taken along the contour L and I_2 the sum of the integrals along the two parts of the contour M.

In the first integral I_1 put $xz = \zeta$ and let L′ be the transformed contour. The integral becomes

$$\frac{\iota}{x^{\beta+k}\,2\pi}\int_{\mathrm{L}'} e^{-\zeta}\left[\left(-\frac{x}{\zeta}\right)^k \sum_{n=k}^{\infty} c_n\left(-\frac{\zeta}{x}\right)^n\right](-\zeta)^{k+\beta-1}\,d\zeta.$$

For any assigned finite value of k, however large,

$$\left|\left(-\frac{x}{\zeta}\right)^k \sum_{n=k}^{\infty} c_n\left(-\frac{\zeta}{x}\right)^n\right| = \left|c_k - c_{k+1}\frac{\zeta}{x} + c_{k+2}\left(\frac{\zeta}{x}\right)^2 - \ldots\right|$$

$$< |c_k| + |c_{k+1}|\,l' + |c_{k+2}|\,l'^2 + \ldots$$

This series is absolutely convergent and independent of x or ζ. We may therefore say that

$$\left|\left(-\frac{x}{\zeta}\right)^k \sum_{n=k}^{\infty} c_n\left(-\frac{\zeta}{x}\right)^n\right| < \mathrm{R}_k,$$

where R_k is independent of x or ζ, and is finite when k is finite.

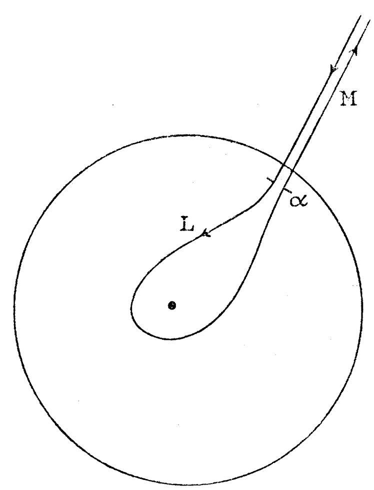
Hence

$$|\mathrm{I}_1| < \frac{1}{|x^{\beta+k}|}\left|\frac{\iota}{2\pi}\right|\left|\int_{\mathrm{L}'}|e^{-\zeta}|\,\mathrm{R}_k\,|(-\zeta)^{k+\beta-1}|\,|d\zeta|\right|.$$

Thus $|x^{\beta+k-1}\mathrm{I}_1|$ can be made as small as we please by taking $|x|$ sufficiently large.

Consider in the next place the integral I_2.

If the original contour cut none of the lines of dissection of the plane, it may be closed up as in the figure. For, as we pass over no poles of the subject of integration, by CAUCHY's theorem we do not alter its value. The contour integral I_2 can therefore be replaced by



$$\frac{\sin\pi\beta}{\pi}\int e^{-xz}z^{\beta-1}\left[f(z) - \sum_{n=0}^{k-1} c_n(-z)^n\right]dz,$$

a line integral taken from the point α along the axis P to infinity.

If we put $z = \alpha + \zeta/x$, we get

$$\mathrm{I}_2 = \frac{\sin\pi\beta}{\pi}e^{-\alpha x}\int_0^{\infty} e^{-\zeta}(\alpha+\zeta/x)^{\beta-1}\left[f(\alpha+\zeta/x) - \sum_{n=0}^{k-1} c_n(-\alpha-\zeta/x)^n\right]d\zeta/x,$$

and the integral is taken along a line for which $\mathfrak{R}(\zeta)$ is positive.

By our original hypothesis the integral is convergent. It is finite for any assigned finite value of k, and when $|x|$ is very large it tends to a finite limit.

Hence $|I_2|$ tends exponentially to zero with $1/|x|$.

Therefore for all finite values of k however large we may take $|x|$ so large that

$$\left|I - \sum_{n=0}^{k-1} c_n/x^{\beta+n}\Gamma(1-\beta-n)\right| |x^{\beta+k-1}|$$

tends to zero as $|x|$ increases.

Therefore I admits the asymptotic expansion

$$\sum_{=0}^{\infty} c_n/x^{\beta+n}\Gamma(1-\beta-n).$$

Inasmuch as $f(z)$ admits $\sum_{n=0}^{\infty} c_n z^n$ as a summable divergent series on the dissected plane, we may say that, for our process for deriving an asymptotic expansion from an integral of the specified type to be possible, the contour C must be such that for all points on and within it $f(z)$ must be representable by the summable divergent series.

Part I.

The Function $G(x;\theta) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n+1)(n+\theta)}$.

§ 6. The function $G(x;\theta)$ is a particular case of the function $G_\beta(x;\theta)$ which will be considered in Part III. It can be discussed by more simple methods than are used in the more general case, and some of the formulæ can only be deduced from the latter by employing the calculus of limits. I give here a brief summary of results and refer the reader elsewhere* for detailed analysis. We assume that θ is not zero or a negative integer.

I. By considering the contour integral

$$-\frac{1}{2\pi\iota}\int \frac{\Gamma(-s)x^s}{s+\theta}ds$$

we can prove that

$$G(-x;\theta) = -e^{-x}\sum_{n=0}^{k} \frac{(\theta-1)\ldots(\theta-n)}{x^{n+1}} + \frac{(\theta-1)\ldots(\theta-k-1)}{x^{k+1}}G(-x;\theta-k-1).$$

II. If $|\arg x| < \pi$,

$$G(-x;\theta) - x^{-\theta}\Gamma(\theta) = -x^{-\theta}\int_x^{\infty} e^{-y}y^{\theta-1}dy,$$

where the line of integration is straight, tends to infinity in the positive half of the y-plane, does not cut the negative half of the real axis, and avoids the origin.

* See the 'Quarterly Journal of Mathematics,' 1906, vol. 37, pp. 289–313.

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III. If $|\arg x| < \pi$, we have the asymptotic expansion

$$G(-x;\theta) = \Gamma(\theta)\,x^{-\theta} - e^{-x}\sum_{n=0}^{\infty}\frac{(\theta-1)\ldots(\theta-n)}{x^{n+1}}.$$

When $|\arg x|$ is very small, we have the asymptotic expansion

$$G(x;\theta) = e^{x}\sum_{n=0}^{\infty}\frac{(1-\theta)\ldots(n-\theta)}{x^{n+1}}.$$

These expansions are truly asymptotic in the sense of § 3.

IV. The large zeros of $G(x;\theta)$ occur near the positive or negative directions of the imaginary axis.

Part II.

The Function $g(x;\theta)$ defined, when $|x|<1$, by the Taylor's *series* $\sum_{n=0}^{\infty}\frac{x^n}{n+\theta}$.

§ 7. This function is, in Borel's language, the function associated with $G(x;\theta)$. It is a particular case of the more general function considered in Part IV. The detailed analysis is given in the paper to which reference was made in Part I.

I. The function $g(x;\theta)$ can, for all values of x except those which lie on the real axis between 1 and $+\infty$ (the limits included), be represented by the system of integrals

$$\frac{1}{2\pi\iota}\int_{D} G(xz;\theta)\log(-z)\,e^{-z}dz,$$

where D is a contour which encloses the origin and embraces some line in the positive half of the z-plane along which the integral is finite, and where $\log(-z)$ is real when z is real and negative and has a cross cut along this line.

II. We deduce that the only finite singularities of $g(x;\theta)$ must lie on the real axis between $x = 1$ and $x = +\infty$ (the limits included).

III. By using I., coupled with the asymptotic expansion of $G(x;\theta)$, we can show that, if $|\arg x| < \pi/2$, and $|(1-x)/x| < 1$,

$$-g(x;\theta)-\psi(\theta)\,x^{-\theta} = \sum_{n=0}^{\infty}\frac{(\theta-1)\ldots(\theta-n)}{n\,!}\frac{(1-x)^n}{x^{n+1}}\{\log(1-x)-\psi(n+1)\}.$$

This formula gives the nature of the singularity of $g(x;\theta)$ at $x = 1$, and shows that $g(x;\theta)$ has no other singularities in the finite part of the plane.

IV. The function $xg(x;\theta)+x^{1-\theta}\log(1-x)$ satisfies the differential equation

$$x\frac{d\eta}{dx}+(\theta-1)\,\eta = x\,\frac{1-x^{1-\theta}}{1-x}.$$

By considering this equation we may again deduce the expansion, valid when $\mathfrak{R}(x) > \frac{1}{2}$,

$$g(x;\theta) + x^{-\theta}\log(1-x) = \{\psi(1) - \psi(\theta)\}x^{-\theta} + \sum_{n=1}^{\infty} \frac{(1-x)^n}{x^{n+1}} \frac{(\theta-1)\dots(\theta-n)}{n!}\left(\frac{1}{1} + \dots + \frac{1}{n}\right).$$

V. We may equally show that, when $|1-x|$ is sufficiently small,

$$x^{\theta}g(x;\theta) + \log(1-x) = \psi(1) - \psi(\theta) + \log x - \sum_{n=1}^{\infty} S_{n-1}(\theta)(\log x)^n/n!,$$

where $S_{n-1}(\theta)$ is the $(n-1)^{\text{th}}$ simple Bernoullian function of θ of parameter unity.

VI. When θ is not a positive or negative integer or zero, we have, if $|x| > 1$, and $|\arg(-x)| < \pi$,

$$g(x;\theta) = -\sum_{n=1}^{\infty} \frac{1}{x^n(\theta-n)} - (-x)^{-\theta}\frac{\pi}{\sin \pi\theta}.$$

This formula gives the asymptotic value of $g(x;\theta)$ when $|x|$ is very large.

Part III.

The Function $G_{\beta}(x;\theta) = \sum_{n=0}^{\infty} \frac{x^n}{n!(n+\theta)^{\beta}}$.

§ 8. This function is the generalisation of that considered in Part I. We assume that θ is not zero or a negative integer, and further that

$$(n+\theta)^{\beta} = \exp\{\beta \log(n+\theta)\},$$

wherein the absolute value of the imaginary part of $\log(n+\theta)$ is less than π.

If θ be real and negative, this convention fails to define $(n+\theta)^{\beta}$ for those terms for which $(n+\theta) < 0$. In these cases we may assume that the imaginary part of $\log(n+\theta)$ is equal to $+\pi\iota$.

§ 9. Suppose in the first place that $\mathfrak{R}(x) > 0$. Then we have the following lemma* :—

If $|\arg x| < \pi/2$, the integral $\frac{1}{2\pi\iota}\int \frac{x^s\Gamma(-s)}{(s+\theta)^{\beta}}ds$ vanishes when taken along any part of the great circle at infinity for which $\mathfrak{R}(s) > -l$, where l is any finite positive quantity, provided the circle pass between the points $s = n$, n being a positive integer.

The same integral is finite when taken along any parallel to the imaginary axis in the finite part of the plane, which does not pass through finite singularities of the subject of integration.

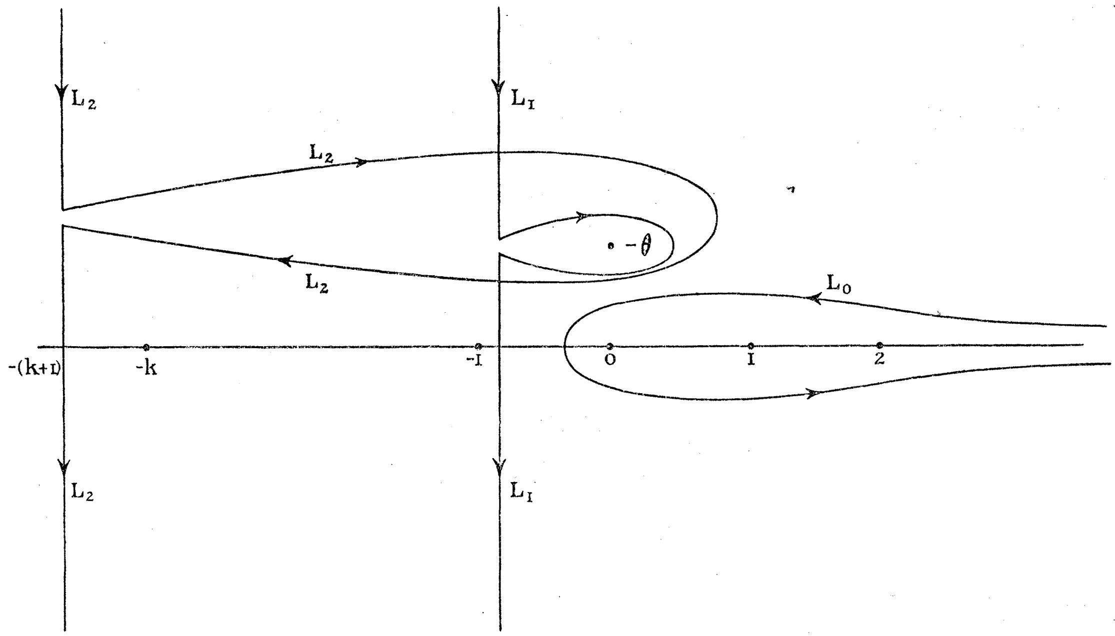
* With this theorem the reader may compare the method used by the author to obtain the asymptotic expansion of the multiple gamma function, 'Cambridge Philosophical Transactions,' vol. 19, §§ 55–57.

The second part of the theorem is true, since when $s = u + \iota v$ and $|v|$ is very large, $|\Gamma(-s)|$ behaves like $\exp\left\{-\frac{\pi}{2}|v|\right\}$.

The first part follows from this fact in combination with the asymptotic expansion of $\Gamma(-s)$ for complex values of s.

§ 10. If L_1 be a contour parallel to the imaginary axis and cutting the real axis between $s = 0$ and $s = -1$, the contour, if necessary, having a loop to ensure that $-\theta$ is to its left as in the figure, then

$$G_\beta(-x; \theta) = -\frac{1}{2\pi\iota}\int_{L_1}\frac{x^s\Gamma(-s)}{(s+\theta)^\beta}\,ds.$$



For by the lemma we may bend the contour round until it becomes the contour L_0 of the figure.

The residue of the subject of integration at $s = n$ is

$$\underset{\epsilon=0}{\mathrm{Lt}} - \frac{x^n}{(n+\theta)^\beta}\epsilon\Gamma(-n-\epsilon) = \frac{(-x)^n}{(n+\theta)^\beta\,\Gamma(n+1)}.$$

Hence by CAUCHY's theorem we have the proposition stated.

§ 11. Let L_2 be a contour parallel to the imaginary axis (except for a loop round $-\theta$) which cuts the real axis in $s = -\lambda$ between $s = -k$ and $s = -(k+1)$, then

$$G_\beta(-x; \theta) = -\frac{1}{2\pi\iota}\int_{L_2}\frac{x^s\Gamma(-s)}{(s+\theta)^\beta}\,ds.$$

This follows from CAUCHY's theorem combined with the second part of the lemma.

§ 12. The integral along the straight parts of the contour may be denoted by I_k.

It is evident that $|x^k I_k|$ tends to zero as $|x|$ tends to infinity for any finite value of k.

In the remaining part of the integral put $s+\theta = -y$ and we find

$$G_\beta(-x;\theta) = -\frac{x^{-\theta}}{2\pi\iota}\int_{C_\lambda}(-y)^{-\beta}x^{-y}\Gamma(y+\theta)\,dy + I_k,$$

the integral being taken round a contour C_λ, which encloses the origin and embraces the positive half of the real axis up to the point $\lambda - \mathfrak{R}(\theta)$.

The poles of $\Gamma(y+\theta)$ are at the points $-\theta-r$, $r = 0, 1, 2, \ldots \infty$.

Hence, within a circle of radius equal to the minimum value of $|\theta+r|$, it admits the convergent expansion

$$\sum_{r=0}^{\infty}\frac{\Gamma^{(r)}(\theta)\,y^r}{r!}.$$

I will show that $G_\beta(-x;\theta)$ admits the asymptotic expansion

$$\sum_{r=0}^{\infty}\frac{\iota}{2\pi}x^{-\theta}\int\frac{(-)^r\Gamma^{(r)}(\theta)}{r!}(-y)^{r-\beta}e^{-y\log x}\,dy,$$

the integrals being taken round a gamma-function contour which encloses the origin, embraces the real axis, and passes from positive infinity to positive infinity again.

This expansion may evidently be written

$$x^{-\theta}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma^{(r)}(\theta)}{r!\,\Gamma(\beta-r)(\log x)^{r-\beta+1}}.$$

We have, if m be a finite positive integer,

$$G_\beta(-x;\theta) - I_k - x^{-\theta}\sum_{r=0}^{m}\frac{(-)^r\Gamma^{(r)}(\theta)}{r!\,\Gamma(\beta-r)(\log x)^{r-\beta+1}}$$

$$= \frac{\iota}{2\pi}x^{-\theta}\int_{C_\lambda}(-y)^{-\beta}x^{-y}\left[\Gamma(y+\theta) - \sum_{r=0}^{m}\frac{\Gamma^{(r)}(\theta)}{r!}y^r\right]dy - \sum_{r=0}^{m}\frac{\sin\pi\beta}{\pi}\int_{\lambda-\mathfrak{R}(\theta)}^{\infty}y^{-\beta+r}x^{\theta-y}\frac{\Gamma^{(r)}(\theta)}{r!}dy\,. \quad (1),$$

the latter integrals being taken along the positive half of the real axis.

If we denote the sum of these integrals by J, we readily see that, for any finite value of m, we can by taking $|x|$ sufficiently large make $|Jx^k|$ as small as we please.

§ 13. We have now to consider the first integral in (1).

Let η be a point on the positive half of the real axis just within the circle of convergence of $\Gamma(y+\theta)$, so that $|\eta| <$ the minimum value of $|\theta+n|$, $n = 0, 1, 2, \ldots \infty$.

Then the first integral in question can be split up into two others, I_1 and I_2 (say). We denote by I_1 the integral round a contour M (say), enclosing the origin and passing from the point η to this point again, η being on the cross-cut which renders the subject of integration uniform. The remaining integral I_2 will be equal to

$$\frac{\sin\pi\beta}{\pi}x^{-\theta}\int_{\eta}^{\lambda-\mathfrak{R}(\theta)}y^{-\beta}x^{-y}\left\{\Gamma(y+\theta) - \sum_{r=0}^{m}\frac{\Gamma^{(r)}(\theta)\,y^r}{r!}\right\}dy.$$

If ${}_mG_k$ be the maximum value of

$$\left|\frac{\sin \pi\beta}{\pi} y^{-\beta}\left\{\Gamma(y+\theta) - \sum_{r=0}^{m} \frac{\Gamma^{(r)}(\theta)\, y^r}{r!}\right\}\right|$$

on this line, we shall have

$$|I_2| < {}_mG_k |x^{-\theta-\eta}| \quad |\lambda - \mathfrak{R}(\theta) - \eta|.$$

Thus for all finite values of k and m, however large, we can take $|x|$ so large that $|x^{\theta+\eta-\epsilon} I_2|$, where $\epsilon > 0$ and as small as we please, tends to zero as nearly as we please.

Finally the integral I_1

$$= \frac{\iota}{2\pi} x^{-\theta} \int_{M} (-y)^{-\beta} x^{-y} \left\{\sum_{r=m+1}^{\infty} \frac{\Gamma^{(r)}(\theta)}{r!} y^r\right\} dy.$$

By the substitution $\eta = y \log x$, we see exactly as in § 5 that $|I_1 x^{\theta} (\log x)^{-\beta+1+m}|$ can be made as small as we please by taking $|x|$ sufficiently large.

Hence

$$|x^{\theta}(\log x)^{-\beta+1+m}| \quad \left|G_\beta(-x;\theta) - x^{-\theta} \sum_{r=0}^{m} \frac{(-)^r \Gamma^{(r)}(\theta)}{r!\,\Gamma(\beta-r)(\log x)^{r-\beta+1}}\right|$$

can for any finite value of m be made as small as we please by taking $|x|$ sufficiently large.

Therefore, provided $\mathfrak{R}(x) < 0$ and θ is not real and negative, we have the asymptotic expansion

$$G_\beta(x;\theta) = (-x)^{-\theta}[\log(-x)]^{\beta-1} \sum_{r=0}^{\infty} \frac{(-)^r \Gamma^{(r)}(\theta)}{\Gamma(r+1)\,\Gamma(\beta-r)\,\{\log(-x)\}^r},$$

the principal value of $\log(-x)$, which is real when x is real and negative, being taken.

§ 14. We proceed now to obtain the asymptotic expansion of

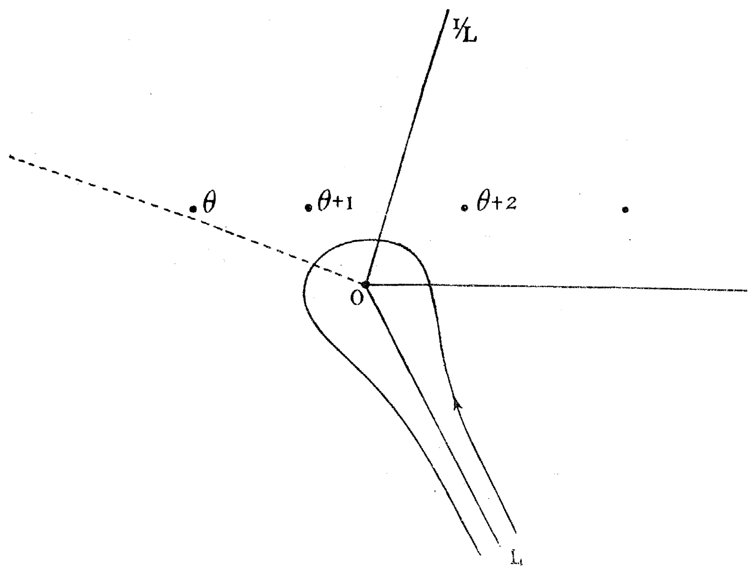
$$G_\beta(x;\theta) = \sum_{n=0}^{\infty} x^n/n!\,(n+\theta)^\beta$$

in the case in which $\mathfrak{R}(x) > 0$.

We will assume that θ is not real and negative. In this case the points θ, $\theta+1$, $\theta+2$, ... all lie within an angle, vertex the origin, which is less than π.

Let the bisector of this angle be the line 1/L, and let L be the image of this line in the real axis. The figure is drawn for the case in which the imaginary part of θ is positive.

Suppose now that $(-z)^{\beta-1} = \exp\{(\beta-1)\log(-z)\}$ when the logarithm is rendered one-valued by a cross-cut along the axis L, and $\log(-z)$ is such that it is real when z is real and negative.



Then*

$$\frac{\iota\Gamma(1-\beta)}{2\pi}\int_{L}(-z)^{\beta-1}e^{-(\theta+n)z}\,dz=(\theta+n)^{-\beta}$$

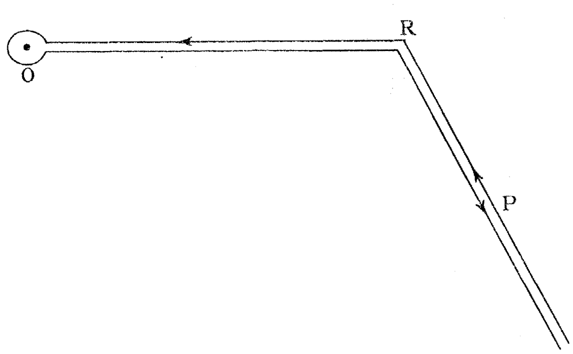
when the integral is taken along a contour, as in the figure, embracing the axis L, and $(\theta+n)^{-\beta}=\exp[-\beta\log(\theta+n)]$, the logarithm being rendered one-valued by a cross-cut along the negative direction of the axis of 1/L, and $\log(\theta+n)$ being real when $\theta+n$ is real and positive. It is a value of $(\theta+n)^{-\beta}$ so defined which we suppose to intervene in the fundamental series by which $G_{\beta}(x;\theta)$ is defined. [*Vide* § 8.]

We now have

$$G_{\beta}(x;\theta)=\frac{\iota\Gamma(1-\beta)}{2\pi}\int_{L}(-z)^{\beta-1}\sum_{n=0}^{\infty}\frac{x^{n}}{n!}e^{-(\theta+n)z}\,dz$$

$$=\frac{\iota\Gamma(1-\beta)}{2\pi}\int_{L}(-z)^{\beta-1}\exp\{-z\theta+e^{-z}x\}\,dz.$$

§ 15. We may deform the original contour L till it has the position of the figure.



It thus consists of a small curcuit round the origin, the real axis from 0+ to R described both ways and a line P from R to ∞ parallel to the original direction of the

* 'M. G. F.,' p. 388. [In this manner reference will be made to the paper cited in § 9.]

axis of the contour which is also described both ways. We suppose that the cross-cut which renders $(-z)^{\beta-1}$ uniform has been deformed with the contour.

The value of the integral in the two directions along the final line P is, putting $z = \mathrm{R}+\zeta$,

$$\mathrm{I}_2 = \frac{1}{\Gamma(\beta)}\int_0^\infty(\mathrm{P})\,(\mathrm{R}+\zeta)^{\beta-1}\exp\{-\mathrm{R}\theta-\zeta\theta+e^{-\mathrm{R}-\zeta}x\}\,d\zeta,$$

the principal value of $(\mathrm{R}+\zeta)^{\beta-1}$, which has a cross-cut along the negative half of the real axis, being taken. The integral is a line integral along P.

On this line $\mathfrak{R}(\zeta) > 0$. Hence the maximum value of the real part of $e^{-\mathrm{R}-\zeta}x$ is $e^{-\mathrm{R}}|x|\mathrm{K}$, where K is a finite quantity independent of R and $|x|$.

Hence

$$|\mathrm{I}_2| < \frac{1}{\Gamma(\beta)}\exp\{e^{-\mathrm{R}}|x|\mathrm{K}-\mathrm{R}\mathfrak{R}(\theta)\}\,|\mathrm{R}^{\beta-1}|$$

$$\times\int_0^\infty(\mathrm{P})\left|\left(1+\frac{\zeta}{\mathrm{R}}\right)^{\beta-1}\right|\,|\exp(-\theta\zeta)|\,|d\zeta|.$$

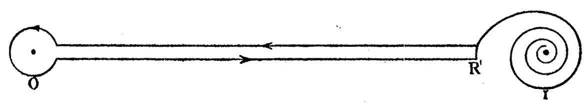
The last integral is convergent and tends to a definite finite value Q as R increases.

When $|x|$ is large let us take $|x| = e^{\mathrm{R}}$. Then

$$|\mathrm{I}_2| < \frac{1}{\Gamma(\beta)}\exp \mathrm{K}\,\{\log|x|\}^{\mathfrak{R}(\beta)-1}\,\frac{\mathrm{Q}}{|x|^{\mathfrak{R}(\theta)}}.$$

When $\mathfrak{R}(\theta) > 0$, $|\mathrm{I}_2|$ will obviously tend to zero as R tends to infinity, and this is true of $|e^{-x}\mathrm{I}_2|$ for all values of θ which are not real and negative, if $\mathfrak{R}(x) > 0$.

§ 16. We have now to consider the value of the original integral along the contour which consists of the small circuit round the origin and the real axis described both ways from $0+$ to R. We denote this integral by I_1. Make the transformation $1-y = e^{-z}$. Then corresponding to the original modified contour we have a new contour Q as in the figure. This consists of a small circuit round the origin and the



real axis described both ways from $0+$ to R′, where $\mathrm{R}' = 1-e^{-\mathrm{R}}$. The former line P from R to ∞ becomes an infinite spiral from R′ round the point 1 up to this point, the whole spiral being contained within a circle of radius $e^{-\mathrm{R}}$.

We now have

$$\mathrm{I}_1 = e^x\,\frac{\iota\Gamma(1-\beta)}{2\pi}\int_{\mathrm{Q}}[\log(1-y)]^{\beta-1}(1-y)^{\theta-1}e^{-xy}\,dy,$$

the integral being taken round the origin from R′ back to R′. The many-valued functions which intervene in the subject of integration are such that, when $|y| < 1$,

$$[\log(1-y)]^{\beta-1}(1-y)^{\theta-1} = (-y)^{\beta-1}\sum_{n=0}^{\infty}c_n(-y)^n \quad . \quad . \quad . \quad . \quad . \quad (1),$$

the principal value of $(-y)^{\beta-1}$, which is real when y is real and negative and has a cross-cut along the line joining 0 and 1, being taken.

Evidently $c_0 = 1$. For the evaluation of the other coefficients of the expansion the reader may refer to a paper by FÉRAND.*

Now

$$e^x \frac{\iota\Gamma(1-\beta)}{2\pi} \int_Q c_n (-y)^{n+\beta-1} e^{-xy}\, dy$$

$$= e^x \frac{\iota\Gamma(1-\beta)}{2\pi} \int_G c_n (-y)^{n+\beta-1} e^{-xy}\, dy - \frac{(-)^n}{\Gamma(\beta)} \int_{1-e^{-R}}^{\infty} c_n y^{\beta+n-1} e^{x(1-y)}\, dy,$$

where G is a gamma-function contour enclosing the origin and embracing the positive half of the real axis and the second line integral is taken along the real axis.

Therefore

$$e^x \frac{\iota\Gamma(1-\beta)}{2\pi} \int_Q c_n (-y)^{n+\beta-1} e^{-xy}\, dy = e^x \frac{c_n \Gamma(1-\beta)}{\Gamma(1-\beta-n)\, x^{\beta+n}} - J_n,$$

where $|J_n| < K'_n |\exp\{e^{-R}x\}| < K'_n |\exp\cos(\arg x)|$, and K'_n is a finite positive quantity for all finite values of n.

Hence

$$I_1 - e^x \sum_{n=0}^{k-1} \frac{c_n \Gamma(1-\beta)}{\Gamma(1-\beta-n)\, x^{n+\beta}} = e^x \frac{\iota\Gamma(1-\beta)}{2\pi} \int_Q \left[\sum_{n=k}^{\infty} c_n (-y)^{\beta+n-1}\right] e^{-xy}\, dy - \sum_{n=0}^{k-1} J_n.$$

But as in § 5 we see that

$$\left| x^{\beta+k-1} \int_Q \left[\sum_{n=k}^{\infty} c_n (-y)^{\beta+n-1}\right] e^{-xy}\, dy \right|$$

can be made as small as we please for any assigned value of k by taking $|x|$ sufficiently large.

Hence, when $\mathfrak{R}(x) > 0$, $e^{-x} I_1$ admits the asymptotic expansion

$$\sum_{n=0}^{\infty} \frac{c_n \Gamma(1-\beta)}{\Gamma(1-\beta-n)\, x^{\beta+n}}.$$

Reverting to the value which we obtained as the quantity greater than $|I_2|$, we see that $e^{-x} G_\beta(x;\theta)$ admits, when $\mathfrak{R}(x) > 0$, the same asymptotic expansion.

§ 17. In the foregoing investigation θ may have any finite value while not real and negative (zero included).

But the expansion is valid even if θ be real and negative, provided it be not a negative integer.

* FÉRAND, "Bordeaux Procès Verbaux," 1896–97, pp. 93–97; quoted in the 'Fortschritte der Mathematik,' vol. 29, p. 375.

For if θ lie between $-k$ and $-(k+1)$, k being a positive integer, we consider $G_\beta(x\,;\,\theta+k)$.

By the theorem just proved, $G_\beta(x\,;\,\theta+k)$ admits an asymptotic expansion.

Also $G_\beta(x\,;\,\theta+k) = \dfrac{d^k}{dx^k}\left[G_\beta(x\,;\,\theta) - \sum\limits_{n=0}^{k-1} \dfrac{x^n}{\Gamma(n+1)(n+\theta)^\beta}\right]$.

Hence, since by a theorem due to POINCARÉ, we may integrate an asymptotic series, the general result follows.*

Finally we see that, if $\Re(x) > 0$, and θ be not zero or a negative integer, the function $G_\beta(x\,;\,\theta)$ admits the asymptotic expansion

$$\frac{e^x}{x^\beta} \sum_{n=0}^{\infty} \frac{c_n \Gamma(1-\beta)}{\Gamma(1-\beta-n)\, x^n}.$$

The coefficients c_n are determined by the expansion

$$\left[\frac{\log(1-y)}{-y}\right]^{\beta-1}(1-y)^{\theta-1} = \sum_{n=0}^{\infty} c_n(-y)^n, \quad \text{valid where } |y| < 1.$$

§ 18. But when $\Re(x) < 0$, we have also obtained an asymptotic expansion, and the restriction that θ shall not be real and negative can be replaced by the restriction that θ shall not be zero or a negative integer.

Combining the two results we see that, when $|x|$ is large and θ not zero or a negative integer, the behaviour of $G_\beta(x\,;\,\theta)$ is given by the sum of the two asymptotic expansions

$$\frac{e^x}{x^\beta} \sum_{n=0}^{\infty} \frac{c_n \Gamma(1-\beta)}{\Gamma(1-\beta-n)\, x^n} + (-x)^{-\theta} [\log(-x)]^{\beta-1} \sum_{n=0}^{\infty} \frac{(-)^n \Gamma^{(n)}(\theta)}{\Gamma(n+1)\,\Gamma(\beta-n)\{\log(-x)\}^n} \qquad \text{(A)}.$$

This double expansion is valid for all values of $\arg x$ between $-\pi$ and π, except possibly those for which $|\arg x| = \dfrac{\pi}{2}$. It is, as we shall see later, true even in these cases.

§ 19. When $\beta=1$, the function reduces to $G(x\,;\,\theta) = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{\Gamma(n+1)(n+\theta)}$, which was previously considered.

The c's are now determined by the expansion $(1-y)^{\theta-1} = \sum\limits_{n=0}^{\infty} c_n(-y)^n$.

Therefore $c_n = (\theta-1)(\theta-2)\ldots(\theta-n)/n\,!$

The asymptotic formula (A) therefore reduces to

$$\frac{e^x}{x} \sum_{n=0}^{\infty} \frac{(-)^n(\theta-1)\ldots(\theta-n)}{x^n} + (-x)^{-\theta}\Gamma(\theta),$$

which was the result previously obtained (§ 6, III.).

* The matter does not seem of sufficient importance for an elaborate proof. I may, however, refer the reader to Mr. HARDY'S paper, p. 419 (*loc. cit.*, § 2).

§ 20. We have obtained the asymptotic expansion of $G_\beta(x;\theta)$ in the two cases when $\mathfrak{R}(x) > 0$ and when $\mathfrak{R}(x) < 0$ by separate methods. By this, however, we are left in doubt as to the behaviour of the function on the imaginary axis. We proceed now to obtain the two expansions simultaneously.

We shall limit ourselves to the case $\mathfrak{R}(\theta) > 0$, as the result can then be extended to all values of θ, except those which are zero or a negative integer.

As previously, we have

$$G_\beta(x;\theta) = \frac{\iota\Gamma(1-\beta)}{2\pi}\int_L (-z)^{\beta-1}\exp\{-z\theta+e^{-z}x\}\,dz,$$

where now, since $\mathfrak{R}(\theta) > 0$, the contour L can be taken to embrace the positive half of the real axis.

Hence

$$G_\beta(x;\theta) = \frac{\iota\Gamma(1-\beta)}{2\pi}\int (-z)^{\beta-1}\exp\{-z\theta+e^{-z}x\}\,dz + \frac{1}{\Gamma(\beta)}\int_\eta^\infty z^{\beta-1}\exp\{-z\theta+e^{-z}x\}\,dz.$$

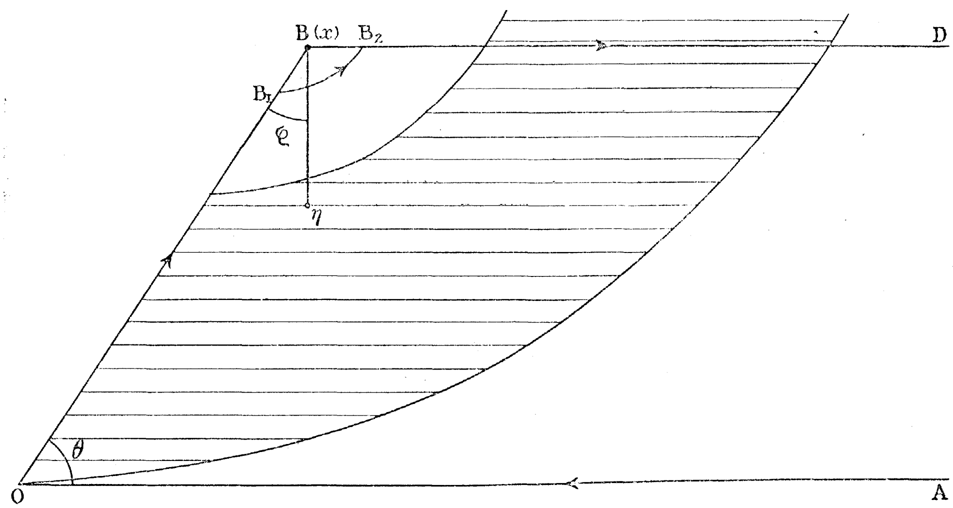
The first integral is taken round a circle of radius η surrounding the origin, beginning and ending on the positive half of the real axis, which is a cross-cut to render $(-z)^{\beta-1}$ uniform. The second integral is taken along the real axis.

If $\mathfrak{R}(\beta) > 0$, the first integral vanishes as η approaches zero. For simplicity we consider only this case. The limitation simplifies the statement of the proof: its absence does not affect the argument.

By the substitution $1-y/x = e^{-z}$, we now obtain

$$\Gamma(\beta)\,G_\beta(x;\theta) = \frac{1}{x}\int_0^x\left[-\log\left(1-\frac{y}{x}\right)\right]^{\beta-1}\left(1-\frac{y}{x}\right)^{\theta-1}e^{x-y}\,dy,$$

the integral being taken along the straight line from O, the origin, to B, the point x.



Take now the integral

$$\int\left[-\log\left(1-\frac{y}{x}\right)\right]^{\beta-1}\left(1-\frac{y}{x}\right)^{\theta-1}e^{x-y}\,dy$$

round the contour AOB_1B_2D of the figure. This contour consists of the real axis from A $(+\infty)$ to O, the origin, the line OB_1, a circular arc B_1B_2 of radius ϵ round B as centre, and the line B_2D, which passes to infinity parallel to OA.

If we take throughout that value of $\left[-\log\left(1-\frac{y}{x}\right)\right]^{\beta-1}\left(1-\frac{y}{x}\right)^{\theta-1}$ which is one-valued on the plane dissected by a cross-cut drawn away from the origin from B to infinity, and which is therefore represented in this region by the series (summable and divergent when $|y|>|x|$) $\left(\frac{y}{x}\right)^{\beta-1}\sum_{n=0}^{\infty} c_n\left(-\frac{y}{x}\right)^n$, we may use CAUCHY's theorem.

We thus see that

$$G_\beta(x\,;\theta)=\frac{1}{x\Gamma(\beta)}\int_0^\infty\left[-\log\left(1-\frac{y}{x}\right)\right]^{\beta-1}\left(1-\frac{y}{x}\right)^{\theta-1}e^{x-y}\,dy$$
$$-\frac{1}{x\Gamma(\beta)}\int_\epsilon^\infty\left[-\log\left(-\frac{\eta}{x}\right)\right]^{\beta-1}\left(-\frac{\eta}{x}\right)^{\theta-1}e^{-\eta}\,d\eta+I_3.$$

The integral I_3 is the integral round the arc B_1B_2. It vanishes in the limit when $\epsilon=0$, since $\mathfrak{R}(\theta)>0$. The two line integrals are taken along OA and B_2D respectively parallel to the positive half of the real axis. In the second integral we have made the substitution $\eta=y-x$.

The first integral by the general theorem of § 5 admits the asymptotic expansion

$$\frac{e^x}{x^\beta}\sum_{n=0}^{\infty}\frac{(-)^n c_n\Gamma(\beta+n)}{\Gamma(\beta)\,x^n}.$$

§ 21. We proceed to consider the second integral.

We have to seek to find on BD the value of $\log[-\log(-\eta/x)]$ which on OB admitted the expansion

$$\log\left[-\log\left(1-\frac{y}{x}\right)\right]=\log\frac{y}{x}+(\ldots)\frac{y}{x}+\ldots,$$

and which is represented by the continuation of this summable divergent series.

Let $x=re^{\iota\theta}$, so that, its principal value being taken, $\log(-x)=\log r+\iota(\theta-\pi)$ θ being the angle of the figure. Let $\eta=\rho e^{\iota(\phi+\theta-\pi)}$, so that ϕ is the angle given in the figure. We assume that r is large, and consider the shaded area bounded by $\rho=1$ and $\rho=r$.

When $-\eta/x$ is real and positive, $|\eta|$ being less than $|x|$, $\log[-\log(-\eta/x)]$ is real. We will show that, for values of η on B_2D within the shaded area, we must take

$$\log\left[-\log\left(-\frac{\eta}{x}\right)\right]=\log\{\log r+\iota(\theta-\pi)\}+\log\left[1-\frac{\log\rho+\iota(\theta+\phi-\pi)}{\log r+\iota(\theta-\pi)}\right]\quad . \quad (2),$$

where, within the shaded area, the final logarithm is such that

$$\exp\left[(\beta-1)\log\left\{1-\frac{\log\rho+\iota(\theta+\phi-\pi)}{\log r+\iota(\theta-\pi)}\right\}\right]$$

admits the expansion

$$\sum_{n=0}^{\infty} \frac{(-)^n \Gamma(\beta)}{\Gamma(\beta-n)\, n!} \left[\frac{\log \rho + \iota(\theta+\phi-\pi)}{\log r + \iota(\theta-\pi)}\right]^n.$$

Evidently the only trouble which arises is with the specification of the imaginary parts of the logarithms. Now when r is large and $r > \rho > 1$, the imaginary part of $\log[-\log(-\eta/x)] = \log[\log r/\rho - \iota\phi]$ is $-\tan^{-1}\{\phi/(\log r/\rho)\}$, the principal value of the inverse tangent which lies between 0 and $\pi/2$ being taken.

The imaginary part of $\log(\log r + \iota\overline{\theta-\pi})$ is similarly $-\tan^{-1}\{(\pi-\theta)/\log r\}$, the inverse tangent again lying between 0 and $\pi/2$.

Also the imaginary part of $\log\left[1 - \frac{\log \rho + \iota(\theta+\phi-\pi)}{\log r + \iota(\theta-\pi)}\right]$ when the value is taken which is represented within the shaded area by the series

$$-\sum_{n=0}^{\infty} \frac{1}{n} \left[\frac{\log \rho + \iota(\theta+\phi-\pi)}{\log r + \iota(\theta-\pi)}\right]^n$$

is also negative and lies between 0 and $\pi/2$.

When r is large and ρ is just greater than unity, all three imaginary parts are very small. The equality (2) is therefore established.

§ 22. We see then that within that part of BD which is within the shaded area

$$\left[-\log\left(-\frac{\eta}{x}\right)\right]^{\beta-1} = [\log(-x)]^{\beta-1} \left\{\sum_{n=0}^{\infty} \frac{(-)^n \Gamma(\beta)}{\Gamma(\beta-n)\, n!} \left[\frac{\log \eta}{\log(-x)}\right]^n\right\}$$

the principal values of $\log \eta$ and $\log(-x)$ being taken. And on the remainder of BD between ϵ and ∞ the expansion continues to hold as one which is divergent but summable. [An exceptional case occurs when x is real and negative, when a slight modification of the contour must be taken to avoid the point O.]

Therefore, by the fundamental theorem of § 5, we have, when $\mathfrak{R}(\theta) > 0$, for the asymptotic expansion of the second integral,

$$\sum_{n=0}^{\infty} (-x)^{-\theta} \int_{\epsilon}^{\infty} e^{-y} y^{\theta-1} \sum_{n=0}^{\infty} \frac{(-)^n [\log y]^n}{\Gamma(\beta-n)\,\Gamma(n+1)\,[\log(-x)]^{n+1-\beta}}\, dy$$

$$= (-x)^{-\theta} [\log(-x)]^{\beta-1} \sum_{n=0}^{\infty} \frac{(-)^n \Gamma^{(n)}(\theta)}{\Gamma(n+1)\,\Gamma(\beta-n)\,[\log(-x)]^n}.$$

Finally, therefore, if $\mathfrak{R}(\theta) > 0$, we have $G_\beta(x;\theta)$ represented by the sum of the two asymptotic expansions

$$\frac{e^x}{x^\beta} \sum_{n=0}^{\infty} \frac{(-)^n c_n \Gamma(\beta+n)}{\Gamma(\beta)\, x^n} + (-x)^{-\theta} [\log(-x)]^{\beta-1} \sum_{n=0}^{\infty} \frac{(-)^n \Gamma^{(n)}(\theta)}{n!\,\Gamma(\beta-n)\,[\log(-x)]^n}.$$

This result is valid for all values of arg x, with the assigned prescription for $\log(-x)$.

The Asymptotic Behaviour of $G_\beta(x;\theta)$, *when* $\mathfrak{R}(x) > 0$, *and* β *tends to Infinity.*

§ 23. The previous asymptotic expansions give us no indication of the behaviour of $G_\beta(x;\theta)$ when β grows indefinitely, and $\mathfrak{R}(x) > 0$. For although we have seen that, so long as β is finite,

$$G_\beta(x;\theta) = e^x \frac{J(x)}{x^\beta},$$

where $|J(x)|$ tends to a definite finite limit when $|x|$ grows indefinitely, yet as β tends to infinity it is possible that $|J(x)|$ will also grow indefinitely. This possibility must now be examined.

§ 24. We base our investigation on the properties of the function

$$S(s) = \sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+1)(n+\theta)^\beta}.$$

The series is convergent if $\mathfrak{R}(\beta+s) > 0$. For, if $\sigma = -s-1$, the general term is equal to

$$\frac{\Gamma(n+\sigma+1)}{\Gamma(n+1)(n+\theta)^\beta} = \frac{\Gamma(\sigma+1)\left\{\left(1+\frac{\sigma}{1}\right)e^{-\frac{\sigma}{1}}\right\}\cdots\left\{\left(1+\frac{\sigma}{n}\right)e^{-\frac{\sigma}{n}}\right\}}{\exp\left\{-\sigma\left(\frac{1}{1}+\cdots+\frac{1}{n}\right)\right\}(n+\theta)^\beta}$$

$$= \frac{\exp\left[\sigma\left\{\frac{1}{1}+\cdots+\frac{1}{n}\right\}\right]e^{-\gamma\sigma}}{\prod_{r=n+1}^{\infty}\left[\left(1+\frac{\sigma}{r}\right)e^{-\frac{\sigma}{r}}\right](n+\theta)^\beta}$$

$$= \frac{\exp\theta_n}{\prod_{r=n+1}^{\infty}\left[\left(1+\frac{\sigma}{r}\right)e^{-\frac{\sigma}{r}}\right]n^{\beta-\sigma}\left(1+\frac{\theta}{n}\right)^\beta},$$

where θ_n tends to zero and n tends to infinity.

Thus, however large the imaginary part of σ may be (even if it is infinite), the series will be absolutely convergent provided $\mathfrak{R}(\beta-\sigma-1) > 0$, that is to say, provided $\mathfrak{R}(\beta+s) > 0$.

The following argument would have been more simple; it would not however have brought out so clearly that the imaginary part of σ may be infinite.

If u_n denote the nth term of the series,

$$\frac{u_{n+1}}{u_n} = \frac{n-s}{n+1}\left(\frac{n+\theta}{n+\theta+1}\right)^\beta = 1-(\beta+s+1)/n+(\ldots)/n^2+\ldots$$

Hence, by a known theorem, the series is convergent if $\mathfrak{R}(\beta+s+1) > 1$.

§ 25. *We will now show that, if* $\mathfrak{R}(\beta+s) > 0$, *and* $\mathfrak{R}(\theta) > 0$,

$$\frac{S(s)}{\Gamma(-s)} = \frac{1}{\Gamma(\beta)} \int_0^\infty y^{\beta-1} e^{-\theta y} (1-e^{-y})^s \, dy,$$

the integration being taken along the positive half of the real axis.

Let $(1-e^{-y})^s = \sum_{n=0}^{N-1} \frac{(-s)\ldots(-s+n-1)}{1.2\ldots n} e^{-ny} + \mathfrak{R}_N$.

Then, if $y \geqq 0$, and $\mathfrak{R}(s) > 0$, and, as before, $\sigma = -s-1$,

$$|\mathfrak{R}_N| = \left| \sum_{n=N}^{\infty} \frac{(-s)\ldots(-s+n-1)}{n!} e^{-ny} \right| \leqq \sum_{n=N}^{\infty} \left| \frac{\prod_{r=1}^{n} \left[\left(1+\frac{\sigma}{r}\right) e^{-\frac{\sigma}{r}}\right] e^{-ny}}{\exp\left\{-\sigma\left(\frac{1}{1}+\ldots+\frac{1}{n}\right)\right\}} \right|.$$

Now $\left| e^{\gamma\sigma} \prod_{r=1}^{n} \left[\left(1+\frac{\sigma}{r}\right) e^{-\frac{\sigma}{r}}\right] \right|$ tends to a definite finite limit as n tends to infinity. Let μ be its maximum value when $n \geqq N$.

Then $|\mathfrak{R}_N| \leqq \mu \sum_{n=N}^{\infty} e^{-ny} |n^{\sigma(1+\epsilon_n)}|$, where ϵ_n tends to zero as n tends to infinity.

Hence, if $y \geqq 0$, $|\mathfrak{R}_N| \leqq K e^{-Ny}$, if $\mathfrak{R}(s) > 0$, where K is a definite finite quantity independent of N and y.

Hence, if $I_N = \int_0^\infty y^{\beta-1} e^{-\theta y} \mathfrak{R}_N(y) \, dy$,

$$|I_N| \leqq K \int_0^\infty y^{\mathfrak{R}(\beta-1)} e^{-y\{N+\mathfrak{R}(\theta)\}} \, dy.$$

Thus I_N tends to zero as N tends to infinity if $\mathfrak{R}(\beta) > 0$ and $\mathfrak{R}(s) > 0$.

Now

$$\frac{1}{\Gamma(\beta)} \int_0^\infty y^{\beta-1} e^{-\theta y} (1-e^{-y})^s \, dy = \frac{1}{\Gamma(\beta)} \int_0^\infty y^{\beta-1} e^{-\theta y} \sum_{n=0}^{N-1} \frac{(-s)\ldots(-s+n-1)}{n!} e^{-ny} + I_N$$

$$= \frac{1}{\Gamma(-s)\,\Gamma(\beta)} \sum_{n=0}^{N-1} \int y^{\beta-1} \frac{\Gamma(n-s)}{n!} e^{-(n+\theta)y} \, dy + I_N$$

$$= \frac{1}{\Gamma(-s)} \sum_{n=0}^{N-1} \frac{\Gamma(n-s)}{n!\,(n+\theta)^\beta} + I_N \text{ if } \mathfrak{R}(\beta) > 0.$$

Make now N tend to infinity, and we see that, if $\mathfrak{R}(\beta) > 0$ and $\mathfrak{R}(s) > 0$,

$$S(s) = \frac{\Gamma(-s)}{\Gamma(\beta)} \int_0^\infty y^{\beta-1} e^{-\theta y} (1-e^{-y})^s \, dy.$$

But both sides of this equality are continuous and finite except for the poles of $\Gamma(-s)$, if $\mathfrak{R}(\theta) > 0$ and $\mathfrak{R}(\beta+s) > 0$. Therefore the equality holds under this limitation.

§ 26. *We will next show that, if* $s = u+\iota v$ *and* $\mathfrak{R}(\beta+s)>0$, *when* $|v|$ *is very large,* $|S(s)|<Ke^{-\frac{1}{2}\pi|v|}$, *where* K *is a finite quantity independent of* v.

For, if $\mathfrak{R}(\theta)>0$,

$$\left|\frac{S(s)}{\Gamma(-s)}\right| = \left|\frac{1}{\Gamma(\beta)}\int_0^\infty y^{\beta-1}e^{-\theta y}(1-e^{-y})^{u+\iota v}dy\right| < \left|\frac{1}{\Gamma(\beta)}\right|\int_0^\infty |y^{\beta-1}e^{-\theta y}|(1-e^{-y})^u dy < K,$$

where K is a definite finite quantity independent of v.

But, when $|v|$ tends to infinity, $|\Gamma(-s)|/e^{-\frac{1}{2}\pi|v|}$ tends to a definite finite limit. We therefore have the given theorem if $\mathfrak{R}(\theta)>0$.

When $\mathfrak{R}(\theta)\not>0$, we can, if $|\theta|$ be finite, find a finite number N such that $\mathfrak{R}(\theta+N)>0$. We can write down a modified integral which shall express all but the first N terms of $S(s)$. The argument used above will hold for the modified integral; the proposition to be proved is evidently true for the first N terms of $S(s)$. And therefore we may establish that the theorem is true under the sole limitation $\mathfrak{R}(\beta+s)>0$, θ not being zero or a negative integer.

§ 27. We will next show that, *if* C_1 *be a contour in the finite part of the plane parallel to the imaginary axis and cutting the real axis in a point for which* $\mathfrak{R}(s)>-\mathfrak{R}(\beta)$, *and if* $\mathfrak{R}(x)>0$, $e^{-x}G_\beta(x;\theta) = -\frac{1}{2\pi\iota}\int_{C_1}S(s)\,x^s ds.$

In the first place we see that this integral is finite. For the series for $S(s)$ is convergent if $\mathfrak{R}(s+\beta)>0$, and as $|v|$ tends to infinity, $|S(s)x^s|<K|x^{\mathfrak{R}(s)}|e^{|v\arg x|-\frac{1}{2}\pi|v|}$.

Thus the subject of integration tends exponentially to zero if $|\arg x|<\frac{1}{2}\pi$.

Let now C be a contour embracing the positive half of the real axis and cutting this axis in the same point as C_1. As we are ultimately to make $\mathfrak{R}(\beta)$ very large, we shall assume that C_1 includes the origin.

Evidently by CAUCHY's theory of residues

$$-\frac{1}{2\pi\iota}\int_C \Gamma(n-s)\,x^s ds = \sum_{m=0}^{\infty}\frac{(-)^m x^{n+m}}{m!} = e^{-x}x^n.$$

Hence if ${}_N G_\beta(x;\theta)$ denote the sum of the first (N+1) terms of the series by which $G_\beta(x;\theta)$ is defined,

$$e^{-x}{}_N G_\beta(x;\theta) = -\frac{1}{2\pi\iota}\int_C \sum_{n=0}^{N}\frac{\Gamma(n-s)}{n!(n+\theta)^\beta}x^s ds.$$

Now, if $|\arg x|<\frac{1}{2}\pi$, $|\Gamma(n-s)x^s|$ tends exponentially to zero as n tends to infinity if $\mathfrak{R}(s)>$ a finite negative quantity, and if s be not in the neighbourhood of the poles of $\Gamma(-s)$.

Therefore the previous integral will vanish when taken round an infinite contour for which $\mathfrak{R}(s)>$ a finite negative quantity. Hence

$$e^{-x}G_N(x;\theta) = -\frac{1}{2\pi\iota}\int_{C_1}\sum_{n=0}^{N}\frac{\Gamma(n-s)}{n!(n+\theta)^\beta}x^s ds.$$

Suppose now that N tends to infinity. Each side of the equality tends to a definite finite limit, and we have the given theorem.

§ 28. We will now *show that, if* l *be a positive quantity such that* $|n+\theta| > l$, $(n = 0, 1, 2, \ldots \infty)$, *and if* R *be any finite quantity, however large, such that* $\mathfrak{R}(\beta) > R+\epsilon$, *where* $\epsilon > 0$, $e^{-x}l^{\beta}G_{\beta}(x;\theta) = J(x)/x^{R}$, *where* $|J(x)|$ *can be made as small as we please, for all values of* $\mathfrak{R}(\beta)$ *however large, by sufficiently increasing* $|x|$.

For we have
$$e^{-x}l^{\beta}G_{\beta}(x;\theta) = -\frac{1}{2\pi\iota}\int_{C_1}\sum_{n=0}^{\infty}\frac{\Gamma(n-s)\,l^{\beta}}{n!\,(n+\theta)^{\beta}}x^{s}\,ds.$$

We take the straight contour to cut the real axis in the point $-R-\epsilon'$, where $0 < \epsilon' < \epsilon$.

Putting $s = -R-\epsilon'+\iota v$, the integral becomes
$$x^{-R-\epsilon'}\frac{1}{2\pi}\int_{C_1}\sum_{n=0}^{\infty}\frac{\Gamma(n+R+\epsilon'-\iota v)\,l^{\beta}}{n!\,(n+\theta)^{\beta}}x^{\iota v}\,dv = x^{-R}J(x) \quad \text{(say)}.$$

From this integral we can show that $|J(x)|$ can be made as small as we please by taking $|x|$ sufficiently large. And as $|\beta|$ increases indefinitely $|J(x)|$ can be made as small as we please.

We thus have the theorem enunciated.

It is evident that by studying the singularities of the function $S(s)$ we could obtain anew the asymptotic expansion of $G_{\beta}(x;\theta)$ when $\mathfrak{R}(x) > 0$. This problem I reserve for a future occasion.

PART IV.

The Singularities of $g_{\beta}(x;\theta)$.

§ 29. The function $g_{\beta}(x;\theta)$ defined when $|x| < 1$ by series $\sum_{n=0}^{\infty} x^{n}/(n+\theta)^{\beta}$ can be studied by the methods previously developed.

We assume that θ is not zero or a negative integer, and that β is not equal to zero or a positive integer. When $\beta = 1$, the function becomes $g(x;\theta)$ previously considered in Part II. When β is a positive integer, the function can be derived from the case $\beta = 1$ by differentiation with regard to θ.

On account of the length of this paper I give some theorems relating to this function, leaving the development of the theory for publication elsewhere.*

I. The function $g_{\beta}(x;\theta)$ has a single singularity in the finite part of the plane. This singularity occurs at $x = 1$ and is not an essential singularity. Near $x = 1$, $g_{\beta}(x;\theta)$ is many-valued.

II. The function $g_{\beta}(x;\theta)-g_{\beta}(x;1)\,x^{1-\theta}$ has no singularities in the finite part of the plane, except the singularity due to $x^{1-\theta}$ at the origin, and if $|\log x| < 2\pi$, it admits the expansion
$$x^{-\theta}\sum_{n=0}^{\infty}\frac{(\log x)^{n}}{n!}\{\zeta(\beta-n,\theta)-\zeta(\beta-n,1)\},$$

where $\zeta(s,a)$ is the simple Riemann ζ-function of parameter unity.

* See a forthcoming paper in the 'Proceedings of the London Mathematical Society.'

III. The function $g_\beta(x\,;\,\theta)-\Gamma(1-\beta)(-\log x)^{\beta-1}x^{-\theta}$ is one-valued near $x=1$, and in the vicinity of this point admits the convergent expansion

$$\sum_{n=0}^{\infty}\frac{(x-1)^n}{x^{n+1}}\bar{\zeta}_{n+1}(\beta,\,\theta),$$

where $\bar{\zeta}_{n+1}(\beta,\,\theta)$ denotes the $(n+1)$ ple Riemann ζ-function of equal parameters unity.

IV. If θ be not real, the function

$$g_\beta(x\,;\,\theta)+g_\beta\left(\frac{1}{x}\,;\,-\theta\right)e^{\mp\pi\iota\beta},$$

the negative or positive sign being taken as $\mathrm{I}(\theta)$ is $>$ or <0, is one-valued near $x=1$, and has no singularity at this point.

V. If θ be not zero or a positive or negative integer, $g_\beta(x\,;\,\theta)$ admits, when $|x|$ is very large, the asymptotic expansion

$$-\sum_{n=1}^{\infty}\frac{1}{x^n(\theta-n)^\beta}+\frac{[\log(-x)]^{\beta-1}}{(-x)^\theta}\sum_{n=0}^{\infty}\frac{(-)^n\left(\dfrac{\pi}{\sin\pi\theta}\right)^{(n)}}{n\,!\,\Gamma(\beta-n)[\log(-x)]^n}.$$

This theorem is true when β is a positive integer, in which case the final series is a finite series of β terms.

When θ is a positive integer, a modification of the formula can be deduced.

When $\theta=1$ and β is a positive integer, we obtain SPENCE's formula.*

VI. Similar analysis holds for the more general function defined when $|x|<1$ by

$$\sum_{n=0}^{\infty}\frac{x^n[\log(n+\theta)]^k}{(n+\theta)^\beta}.$$

PART V.

The Function $\mathrm{F}_\beta(x\,;\,\theta)=\sum_{n=0}^{\infty}\dfrac{x^n\chi(n+\theta)}{n\,!\,(n+\theta)^\beta}$.

§ 30. We proceed now to use the previous asymptotic expansions to obtain similar expansions for very general types of integral functions.

Let $\chi(x)$ be a function of x which outside a circle of radius l admits the expansion $\sum_{r=0}^{\infty}b_r/x^r$, so that for values of r greater than an assignable quantity R, $|b_r|<l'^r$, where $l'>l$.

Suppose further that the points $\theta,\,\theta+1,\,\theta+2\ldots$ all lie outside this circle, and that the modulus of the least of them is taken to be $>l'$.

* DE MORGAN, 'Differential and Integral Calculus,' 1842, p. 659.

We proceed to show that the function

$$F_\beta(x;\theta) = \sum_{n=0}^{\infty} \frac{x^n \chi(n+\theta)}{(n+\theta)^\beta \Gamma(n+1)}$$

may be written as the sum of functions

$$\sum_{r=0}^{\infty} b_r G_{\beta+r}(x;\theta),$$

and to obtain its asymptotic expansion when $|\arg x| < \pi/2$.

We have at once

$$F_\beta(x;\theta) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\left[\sum_{r=0}^{k} \frac{b_r}{(n+\theta)^{\beta+r}} + \sum_{r=1}^{\infty} \frac{b_{k+r}}{(n+\theta)^{\beta+k+r}}\right]$$

If then $k > R$, we have

$$\left|\sum_{r=1}^{\infty} \frac{b_{k+r}}{(n+\theta)^{\beta+k+r}}\right| < \sum_{r=1}^{\infty} \frac{l'^{k+r}}{|(n+\theta)^{\beta+k}|\,|(n+\theta)^r|} < \frac{l'^k}{|(n+\theta)^{\beta+k}|} \sum_{r=1}^{\infty} \frac{l'^r}{\mu^r},$$

where μ is the minimum value of $|n+\theta|$.

The modulus of the series is therefore less than

$$\frac{l'^{k+1}}{|(n+\theta)^{\beta+k}|(\mu-l')}, \text{ since } \mu > l'.$$

Hence

$$F_\beta(x;\theta) = \sum_{r=0}^{k} b_r G_{\beta+r}(x;\theta) + J_k,$$

where

$$|J_k| < \sum_{n=0}^{\infty} \frac{|x|^n}{n!} \frac{l'^{k+1}}{|(n+\theta)^{\beta+k}|(\mu-l')}.$$

Hence $|J_k|$ tends to zero as k tends to infinity, since $|n+\theta| > l'$. We therefore have

$$F_\beta(x;\theta) = \sum_{r=0}^{\infty} b_r G_{\beta+r}(x;\theta),$$

the series being absolutely convergent for all finite values of $|x|$.

§ 31. We will next show *that, when* $|\arg x| < \pi/2$,

$$F_\beta(x;\theta) = \frac{e^x}{x^\beta}\left[\sum_{s=0}^{\sigma} \frac{S_s}{\Gamma(1-\beta-s)\,x^s} + \frac{J_\sigma(x)}{x^\sigma}\right],$$

where $S_s = \sum_{m=0}^{s} b_m\, {}_mc_{s-m} \Gamma(1-\beta-m)$ *and* $|J_\sigma(x)|$ *tends to zero as* $|x|$ *tends to infinity. The coefficients* ${}_rc_n$ *are given by the expansion, valid when* $|y| < 1$,

$$\left[\frac{\log(1-y)}{-y}\right]^{\beta+r-1} (1-y)^{\theta-1} = \sum_{n=0}^{\infty} {}_rc_n (-y)^n.$$

We have

$$F_\beta(x;\theta) = \sum_{r=0}^{R} b_r G_{\beta+r}(x;\theta) + \sum_{r=R+1}^{\infty} b_r G_{\beta+r}(x;\theta).$$

Consider the second series. However large r may be, we may put

$$|e^{-x} b_r G_{\beta+r}(x;\theta)| = \frac{\epsilon_r}{|x^{\beta+R}|},$$

where, by § 28, ϵ_r can be made as small as we please by taking $|x|$ sufficiently large.

Hence

$$\left|\sum_{r=R+1}^{\infty} b_r G_{\beta+r}(x;\theta)\, e^{-x} x^{\beta+R}\right| < \sum_{r=1}^{\infty} \epsilon_{R+r} < \eta,$$

where η can be made as small as we please by taking $|x|$ sufficiently large. For we have only to take $|x|$ so large that the quantities ϵ_r form an absolutely convergent series.

Again, by the asymptotic expansion for $G_\beta(x;\theta)$, when $|\arg x| < \pi/2$, we have

$$\sum_{r=0}^{R} b_r G_{\beta+r}(x;\theta) = \frac{e^x}{x^\beta} \sum_{r=0}^{R} \frac{b_r}{x^r}\left[\sum_{n=0}^{N} \frac{{}_r c_n \Gamma(1-\beta-r)}{\Gamma(1-\beta-r-n)\, x^n} + \frac{I_N(x)}{x^N}\right],$$

where the coefficients ${}_r c_n$ are defined as in the enunciation of the present proposition, and where $|I_N(x)|$ can be made as small as we please by taking $|x|$ sufficiently large.

We therefore have

$$\sum_{r=0}^{R} b_r G_{\beta+r}(x;\theta) = \frac{e^x}{x^\beta}\left[\sum_{s=0}^{\sigma} \frac{1}{x^s} \sum_{m=0}^{s} \frac{b_m\, {}_m c_{s-m} \Gamma(1-\beta-m)}{\Gamma(1-\beta-s)} + \frac{J_\sigma(x)}{x^\sigma}\right] \quad . \quad . \quad . \quad (A),$$

where $\sigma \leqq N$ and $|J_\sigma(x)|$ can be made as small as we please by taking $|x|$ sufficiently large.

If, now, we take $R > \sigma$ and combine the two results just obtained, we have the proposition stated.

§ 32. The reader will notice the far-reaching generality of the function whose asymptotic expansion has been obtained. He will also notice that we have shown that from the asymptotic expansion of each of a convergent series of functions we have derived an asymptotic expansion for the function represented by the series of functions.

§ 33. *Let us now consider the asymptotic expansion of* $F_\beta(x;\theta)$ *when* $\mathfrak{R}(x) < 0$.

We have seen that

$$F_\beta(x;\theta) = \sum_{r=0}^{\infty} b_r G_{\beta+r}(x;\theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Also, when $|\arg(-x)| < \pi/2$, we have the asymptotic equality

$$G_{\beta+r}(x;\theta) = (-x)^{-\theta}\{\log(-x)\}^{\beta+r+1}\left\{\sum_{n=0}^{N} \frac{(-)^n \Gamma^{(n)}(\theta)}{\Gamma(\beta+r-n)\Gamma(n+1)[\log(-x)]^n} + \frac{{}_r J_N(x)}{[\log(-x)]^N}\right\},$$

where $|{}_r J_N(x)|$ tends to zero as $|x|$ tends to infinity.

By the previous investigation this equality is seen to hold good for all values of r, however large.

Let $F_R(x)$ denote the sum of the first $(R+1)$ terms of the series (1). Then, asymptotically,

$$F_R(x) = (-x)^{-\theta} \{\log(-x)\}^{\beta-1} \sum_{r=0}^{R} b_r \{\log(-x)\}^r$$
$$\left\{ \sum_{n=0}^{N} \frac{(-)^n \Gamma^{(n)}(\theta)}{\Gamma(\beta+r-n)\Gamma(n+1)\{\log(-x)\}^n} + \frac{{}_rJ_N(x)}{\{\log(-x)\}^N} \right\}.$$

It is at once evident that any attempt to make R tend positively to infinity will introduce a series in ascending powers of $\log(-x)$. This series cannot be asymptotic: it may be convergent, or it may be divergent but summable. In order to investigate its nature, we shall limit ourselves to the case where β is a positive integer. In this case the series proceeding in descending powers of $\log(-x)$ will be finite, and if we take $N = \beta+r-1$, $|{}_rJ_N(x)|$ is less than $1/|x|^l$, when $|x|$ is large, however large l may be.

We have then

$$F_R(x) = (-x)^{-\theta} \{\log(-x)\}^{\beta-1} \sum_{r=0}^{R} \left[\sum_{n=0}^{\beta+r-1} \frac{(-)^n \Gamma^{(n)}(\theta)\, b_r}{\Gamma(\beta+r-n)\Gamma(n+1)\{\log(-x)\}^{n-r}} + b_r\, {}_rJ(x) \right].$$

The double series may be written

$$\left\{ \sum_{s=-\beta+1}^{-1} \sum_{r=0}^{R} + \sum_{s=0}^{R} \sum_{r=s}^{R} \right\} \left[\frac{(-)^{r-s} \Gamma^{(r-s)}(\theta)\, b_r \{\log(-x)\}^s}{\Gamma(\beta+s)\Gamma(r-s+1)} \right].$$

Thus

$$(-x)^{\theta} \{\log(-x)\}^{1-\beta} F_R(x) = \sum_{s=1}^{\beta-1} \frac{[\log(-x)]^{-s}}{\Gamma(\beta-s)} \sum_{r=0}^{R} \frac{(-)^{r+s} \Gamma^{(r+s)}(\theta)\, b_r}{\Gamma(r+s+1)}$$
$$+ \sum_{s=0}^{R} \frac{\{\log(-x)\}^s}{\Gamma(\beta+s)} \sum_{r=0}^{R-s} \frac{(-)^r b_{r+s} \Gamma^{(r)}(\theta)}{\Gamma(r+1)} + \sum_{r=0}^{R} b_r\, {}_rJ(x).$$

§ 34. Suppose now that R tends to infinity. The series $|\sum_{r=0}^{\infty} b_r\, {}_rJ(x)|$ can, even when multiplied by *any* finite positive power of $|x|$, be made as small as we please by taking $|x|$ sufficiently large.

The series $\sum_{r=0}^{\infty} \frac{(-)^r b_r \Gamma^{(r+s)}(\theta)}{\Gamma(r+s+1)}$ is absolutely convergent. For if μ be the minimum value of $|(n+\theta)|$, so that μ is the distance of θ from the nearest of the points $0, -1, -2, \ldots$, we have, if $|x| < \mu$,

$$\Gamma(x+\theta) = \sum_{m=0}^{\infty} \frac{\Gamma^{(m)}(\theta)}{\Gamma(m+1)} x^m.$$

Therefore

$$\underset{m=\infty}{\mathrm{L}t} \left| \left[\frac{\Gamma^{(m)}(\theta)}{\Gamma(m+1)} \right]^{1/m} \right| = \frac{1}{\mu}.$$

Thus, since $\underset{r=\infty}{\mathrm{L}t} |b_r^{1/r}| = l$, the series is absolutely convergent, since $\mu > l$.

Again, where s is sufficiently large, $|b_{r+s}| < l'^{r+s}$, where $l' > l$; and therefore

$$\left| \sum_{r=0}^{R-s} \frac{(-)^r b_{r+s} \Gamma^{(r)}(\theta)}{\Gamma(r+1)} \right| < K l'^s,$$

where K is finite and independent of s for all values of R, however large, provided $\mu > l'$.

Hence, when R tends to infinity,

$$\left| \sum_{s=0}^{R} \frac{\{\log(-x)\}^s}{\Gamma(\beta+s)} \sum_{r=0}^{R-s} \frac{(-)^r b_{r+s} \Gamma^{(r)}(\theta)}{\Gamma(r+1)} \right| < K \sum_{s=0}^{\infty} \left| \frac{\{l' \log(-x)\}^s}{\Gamma(\beta+s)} \right| + L,$$

where L is finite; and thus the series is convergent provided $|\log(-x)|$ is finite.

We therefore have the asymptotic equality

$$F_\beta(x;\theta) = \left\{ \phi\{\log(-x)\} + \sum_{s=1}^{\beta-1} \frac{[\log(-x)]^{-s}}{\Gamma(\beta-s)} \sum_{r=0}^{\infty} \frac{(-)^{r+s} b_r \Gamma^{(r+s)}(\theta)}{\Gamma(r+s+1)} + J(x) \right\} (-x)^{-\theta} \{\log(-x)\}^{\beta-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B).$$

This equality is valid provided $\mathfrak{R}(x) < 0$. The function $J(x)$ is such that its modulus, even when multiplied by $|x|^l$, however large the finite quantity l may be, can be made as small as we please by taking $|x|$ sufficiently large. And $\phi(x)$ denotes the integral function

$$\sum_{s=0}^{\infty} \frac{x^s}{\Gamma(\beta+s)} \sum_{r=0}^{\infty} \frac{(-)^r b_{r+s} \Gamma^{(r)}(\theta)}{\Gamma(r+1)} = \sum_{s=0}^{\infty} \frac{x^s T_s}{\Gamma(\beta+s)} \text{ (say).}$$

The quantity $\log(-x)$ is such that the modulus of its imaginary part is less than $\pi/2$.

We see then that the process which we have employed, even in the case where β is an integer, makes the asymptotic expansion of $F_\beta(x;\theta)$, when $\mathfrak{R}(x) < 0$, depend on that of the integral function of $\log(-x)$ which we have written $\phi\{\log(-x)\}$.

§ 35. *We proceed now to apply the theory of contour integration to this problem.*

We have

$$F_\beta(x;\theta) = \sum_{n=0}^{\infty} \frac{x^n \chi(n+\theta)}{(n+\theta)^\beta \Gamma(n+1)},$$

and $\chi(x)$ admits the expansion $\sum_{n=0}^{\infty} b_n/x^n$ outside a circle of finite radius l.

We may represent $F_\beta(x;\theta)$ by the contour integral

$$-\frac{1}{2\pi\iota} \int_{L_0} \frac{(-x)^s \Gamma(-s) \chi(s+\theta)}{(s+\theta)^\beta} ds,$$

taken round a contour L_0 embracing the positive half of the real axis and enclosing $s = 0, 1, 2, \ldots \infty$, but no other poles of the subject of integration. If $l < |\theta + n|$, $n = 0, 1, 2, \ldots \infty$, the whole circle of convergence of $\chi(s+\theta)$, *i.e.*, a circle of radius l and centre $-\theta$, lies outside the contour L_0. If the inequality does not hold, the contour L_0 has to be indented to include $s = 0, 1, 2, \ldots$, but no singularities of $\chi(s+\theta)$. This is always possible if $\chi(\theta+n)$ is not infinite for any positive integral (including zero) value of n.

Suppose now that $|\arg(-x)| < \pi/2$. Then the integral will vanish along any part of an infinite contour for which $\mathfrak{R}(s)$ is greater than a finite negative quanity k (say).

Hence, if $|\arg(-x)| < \frac{\pi}{2}$,

$$F_\beta(x;\theta) = -\frac{1}{2\pi\iota}\int_{L_2}\frac{(-x)^s\Gamma(-s)\chi(s+\theta)}{(s+\theta)^\beta}\,ds,$$

taken along a contour L_2 which consists of the infinite line $s = -k$, and a loop from a point on this line which includes the singularities of $\chi(s+\theta)$, but none of those of $\Gamma(-s)$.

Hence if we neglect terms of order $1/|x|^k$, we see that, when $|x|$ is large and $|\arg(-x)| < \frac{1}{2}\pi$, we have

$$F_\beta(x;\theta) = -\frac{(-x)^{-\theta}}{2\pi\iota}\int(-y)^{-\beta}\chi(-y)\,e^{-y\log(-x)}\,\Gamma(y+\theta)\,dy\,. \quad . \quad . \quad . \quad (1)$$

taken round a contour which encloses within its bulb the singularities of $\chi(-y)$, but none of the points $-\theta$, $-\theta-1$, $-\theta-2, \ldots$, and which embraces the positive half of the real axis. In this integral the principal value of $\log(-x)$, which is real when x is real and negative, must be taken. It is by evaluating this integral for assigned values of $\chi(-y)$ that we can obtain the asymptotic expansion of $F_\beta(x;\theta)$ when $\mathfrak{R}(x) < 0$.

§ 36. Consider the case when β is an integer, positive or negative. In this case the subject of integration is one-valued. The integral can therefore be taken along a finite contour which encloses the singularities of $\chi(y)$, but not the poles of $\Gamma(y+\theta)$. The residues at the latter points will give rise to a finite number of algebraical terms, *i.e.*, terms which involve algebraical powers of x, if there are any such points within the circle of convergence of $\chi(y)$. Making due allowance for such terms, we may take the integral round a circle just larger than this circle of convergence. This integral when $|x|$ is large will be at most of order $|x^{l-\theta}|$. *The same is true of the more general integral* (1) *when β is not an integer*. We thus get a superior limit to the asymptotic value of $F_\beta(x;\theta)$ when $\mathfrak{R}(x) < 0$ and β is or is not integral.

Further progress must be made by a detailed examination of the singularities of $\chi(y)$ within its circle of convergence.

The Function $f(x) = \sum_{n=0}^{\infty} \frac{x^n e^{\frac{1}{n+\theta}}}{\Gamma(n+1)}$.

§ 37. *To illustrate the general theory which has just been developed, we will discuss the function*

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n e^{\frac{1}{n+\theta}}}{\Gamma(n+1)},$$

in which θ is not zero or a negative integer.

With our previous notation we have $\beta = 0$, $b_m = 1/m!$, and l may be taken as small as we please.

Hence, when $\mathfrak{R}(x) > 0$, we have

$$f(x) = e^x \left\{ \sum_{s=0}^{\sigma} \frac{S_s}{x^s} + \frac{J_\sigma(x)}{x^\sigma} \right\},$$

where $|J_\sigma(x)|$ tends to zero as $|x|$ tends to infinity, and where

$$S_s = \mathop{\mathrm{Lt}}_{\beta=0} \sum_{m=0}^{s} \frac{{}_mc_{s-m}\Gamma(1-\beta-m)}{m!\,\Gamma(1-\beta-s)} = \sum_{m=0}^{s} \frac{(-)^{s-m}\,{}_mc_{s-m}\Gamma(s)}{m!\,\Gamma(m)}.$$

Now the coefficients ${}_mc_n$ are given by the expansion, valid when $|y| < 1$,

$$\{\log(1-y)/(-y)\}^{m-1}(1-y)^{\theta-1} = \sum_{n=0}^{\infty} {}_mc_n(-y)^n.$$

Hence S_s is the coefficient of $(-y)^{s-1}$ in the expansion of

$$\Gamma(s)\left[\sum_{m=0}^{\infty} \frac{(-)^{s-m}\{\log(1-y)\}^{m-1}}{m!\,(m-1)!}\right](1-y)^{\theta-1}$$

in ascending powers of y when $|y| < 1$.

The reader will notice the connection between the series in the square brackets and $J_0\{\log(1-y)\}$, where $J_0(x)$ is BESSEL's function of zero order.

When $\mathfrak{R}(x) > 0$, we have now obtained the asymptotic expansion

$$f(x) = e^x \sum_{s=0}^{\infty} \frac{S_s}{x^s}.$$

§ 38. *Consider next the case when* $\mathfrak{R}(x) < 0$.

Then, if $|\arg(-x)| < \pi/2$, we see that $f(x)$ is equal to

$$\frac{1}{2\pi\iota}\int (-x)^s \Gamma(-s)\, e^{\frac{1}{s+\theta}}\, ds,$$

provided we neglect terms whose modulus when multiplied by $|x|^l$, where l has any finite value, can be made as small as we please by taking $|x|$ sufficiently large. The integral is taken in the positive direction round a small circle enclosing $s = -\theta$.

The integral is equal to

$$(-x)^{-\theta} \sum_{n=0}^{\infty} \frac{[\log(-x)]^n}{n!} K_n,$$

where K_n is the coefficient of $\epsilon^{-(n+1)}$ in the expansion of $\Gamma(\theta-\epsilon)\exp\{1/\epsilon\}$.

We therefore have

$$K_n = \sum_{m=0}^{\infty} \frac{(-)^m \Gamma^{(m)}(\theta)}{m!\,(m+n+1)!}.$$

Hence, when $\mathfrak{R}(x) < 0$,

$$f(x) = (-x)^{-\theta} \sum_{n=0}^{\infty} \frac{[\log(-x)]^n}{n!} \sum_{m=0}^{\infty} \frac{(-)^m \Gamma^{(m)}(\theta)}{m!\,(m+n+1)!} + J(x),$$

where $|J(x)x^l|$ for any finite value of l tends to zero as $|x|$ tends to infinity.

The double series obviously represents an integral function of $\log(-x)$, and may be written

$$\sum_{m=0}^{\infty} \frac{(-)^m \Gamma^{(m)}(\theta)}{m!} \sum_{n=0}^{\infty} \frac{y^n}{n!\,(m+1+n)!}, \text{ where } y = \log(-x).$$

Using the notation ${}_0F_1\{\rho\,;x\}$ for the series

$$\sum_{n=0}^{\infty} \frac{y^n \Gamma(\rho)}{n!\,\Gamma(\rho+n)},$$

the double series may be written

$$\phi(y) = \sum_{m=0}^{\infty} \frac{(-)^m \Gamma^{(m)}(\theta)}{m!\,(m+1)!}\, {}_0F_1\{m+2\,;y\}. \quad \text{(Compare Part X.)}$$

§ 39. *We proceed now to obtain an asymptotic expansion for this integral function of y.*

For this purpose we shall anticipate the asymptotic expansions of the hypergeometric series ${}_0F_1\{\rho\,;y\}$, which will be subsequently developed.

We shall show that asymptotically (§ 51, III.)

$${}_0F_1\{\rho\,;y\} = e^{2\sqrt{y}} \frac{\Gamma(2\rho-1)}{\Gamma(\rho-\frac{1}{2})} \{4\sqrt{y}\}^{1/2-\rho}\, {}_2F_0\left\{\rho-\tfrac{1}{2}, \tfrac{3}{2}-\rho\,; \frac{1}{4\sqrt{y}}\right\},$$

wherein $\mathfrak{R}(y) > 0$ and $\sqrt{y}$ denotes the positive value of the square root when y is real and positive.

The series

$${}_2F_0\{\rho_1, \rho_2\,; x\} = 1 + \frac{\rho_1\rho_2}{1}x + \frac{\rho_1(\rho_1+1)(\rho_2)(\rho_2+1)}{1\,.\,2}x^2 + \ldots$$

The modulus of the error which results from stopping at any term of the given series is less than that of the last term retained when $|y|$ is very large.

We shall also show that however large $\mathfrak{R}(\rho)$ may be, if m be any finite quantity $\leqq \rho$, we may put

$$ {}_0F_1\{\rho\,;\,y\} = e^{2\sqrt{y}}\,\frac{I_\rho(y)}{y^{(m-1)/2}}, $$

where, by taking $|y|$ sufficiently large, we may make $|I_\rho(y)|$ as small as we please; and where $|I_\rho(y)|$ tends to a finite limit as ρ tends to infinity, $|y|$ remaining constant.

We now put

$$ \phi(y) = \left[\sum_{m=0}^{R} + \sum_{m=R+1}^{\infty}\right] \frac{(-)^m\,\Gamma^{(m)}(\theta)}{m!\,(m+1)!}\,{}_0F_1\{m+2\,;\,y\}. $$

The first series in which the summation is taken from 0 to R is equal to

$$ e^{2\sqrt{y}}\left[\sum_{m=0}^{R} \frac{(-)^m\Gamma^{(m)}(\theta)}{m!\,(m+1)!}\,\frac{\Gamma(2m+3)}{\Gamma(m+3/2)}\,[4\sqrt{y}]^{-m-3/2}\cdot{}^{R-m+1}_{\;\;2}F_0\left\{m+3/2,\,-1/2-m\,;\,\frac{1}{4\sqrt{y}}\right\} + \frac{J_1(y)}{y^{R/2+3/4}}\right], $$

where ${}^k_2F_0\{y\}$ denotes that the sum of the first k terms of the series in ascending powers of y is to be taken. The modulus of $J_1(y)$ can be made as small as we please by taking $|y|$ sufficiently large.

The second series

$$ \sum_{m=R+1}^{\infty} \frac{(-)^m\,\Gamma^{(m)}(\theta)}{m!\,(m+1)!}\,{}_0F_1\{m+2\,;\,y\} $$

is equal to

$$ e^{2\sqrt{y}} \sum_{m=R+1}^{\infty} \frac{(-)^m\,\Gamma^{(m)}(\theta)}{m!\,(m+1)!}\,\frac{I_m(y)}{y^{(R+2)/2}}, $$

where $|I_m(y)|$ for all values of m can be made as small as we please by sufficiently increasing $|y|$.

Since $\displaystyle\sum_{m=0}^{\infty} \frac{(-)^m\,\Gamma^{(m)}(\theta)}{m!\,(m+1)!}$ is absolutely convergent, we see that the second series may be written

$$ e^{2\sqrt{y}}\,\frac{J_2(y)}{y^{(R+2)/2}}, $$

when $|J_2(y)|$ tends to zero as $|y|$ tends to infinity.

Finally, therefore, if $\mathfrak{R}(x) < 0$ and $|\arg(-x)| < \pi/2$, $f(x)$ admits the asymptotic equality

$$ (-x)^{-\theta}\exp\{2\sqrt{\log(-x)}\}\,\{4\sqrt{\log(-x)}\}^{-3/2}\left[\sum_{m=0}^{R} \frac{(-)^m\Gamma^{(m)}(\theta)}{m!\,(m+1)!}\,\frac{(2m+2)!}{\Gamma(m+3/2)}\,[4\sqrt{\log(-x)}]^{-m}\right. $$

$$ \left. {}^{R-m+1}_{\;\;2}F_0\left\{m+3/2\,,\,-\tfrac{1}{2}-m\,;\,\frac{1}{4\sqrt{\log(-x)}}\right\} + \frac{J(x)}{\{\log(-x)\}^{R/2}}\right], $$

where $|J(x)|$ can be made as small as we please by taking $|x|$ sufficiently large.

§ 40. We have therefore obtained the nature of the asymptotic expansion of $f(x)$ for the two cases when $\mathfrak{R}(x) > 0$ and when $\mathfrak{R}(x) < 0$.

The integral which was employed in § 35 showed us that in the latter case $|f(x)|$ when $|x|$ is very large is of order less than the order of $|(-x)^{-\theta}|\,|x^\epsilon|$ when $\epsilon > 0$.

We readily see that this is in agreement with the preceding result.

The investigation just concluded has emphasised the fact that the asymptotic expansion of the more general function considered at the beginning of this Part of the memoir demands, when $\mathfrak{R}(x) < 0$, a knowledge of the nature of the finite singularities of $\chi(x)$.

Part VI.

The Function $f_\beta(x;\theta) = \sum_{n=0}^{\infty} \frac{x^n \chi(n+\theta)}{(n+\theta)^\beta}$, *when* $|x| < 1$.

§ 41. The function $f_\beta(x;\theta)$ is defined when $|x| < 1$ by the series $\sum_{n=0}^{\infty} \frac{x^n \chi(n+\theta)}{(n+\theta)^\beta}$, where outside a circle of radius $l < \mu$, where μ is the least of the quantities $|n+\theta|$, $n = 0, 1, 2, \ldots \infty$, $\chi(x)$ admits the absolutely convergent expansion $\sum_{r=0}^{\infty} b_r/x^r$.

The following propositions may be established:—

I. When $|x| < 1$, $f_\beta(x;\theta)$ can be written in the form

$$\sum_{r=0}^{\infty} b_r g_{\beta+r}(x;\theta).$$

II. The function $f_\beta(x;\theta)$ has a single singularity in the finite part of the plane. This singularity occurs at $x = 1$, and is a multiform point.

III. Near $x = 1$, $f_\beta(x;\theta)$ behaves like

$$\frac{\pi}{\sin \pi\beta}(-\log x)^{\beta-1} x^{-\theta} \sum_{r=0}^{\infty} \frac{b_r (\log x)^r}{\Gamma(\beta+r)},$$

if β be not a positive or negative integer (zero included). In fact, the difference between $f_\beta(x;\theta)$ and this expression can, when $\mathfrak{R}(x) > 1/2$, be expanded in the absolutely convergent series

$$\sum_{r=0}^{\infty} b_r \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(\beta+r, \theta).$$

This theorem I first prove for the case when $\mathfrak{R}(\theta) > l' > l$, by means of contour integrals similar to those employed throughout this memoir, and then extend to all values of θ such that $\mu > l$, by means of the difference formulæ for multiple Riemann $\bar{\zeta}$ functions.

IV. Whether β be integral or not, in fact, for all values of β of finite modulus, provided the points $\theta \pm n$, $n = 0, 1, 2, \ldots \infty$ all lie outside the circle of convergence of $\chi(x)$,

$$f_\beta(x;\theta) = -\sum_{n=1}^{\infty} \frac{\chi(\theta-n)}{x^n(\theta-n)^\beta} - \frac{1}{2\pi\iota}\int_{C'} \frac{\pi(-x)^{-\theta-y}\chi(-y)\,dy}{\sin\pi(\theta+y)(-y)^\beta},$$

where C′ is a contour which embraces the positive half of the real axis and includes within its bulb the circle of convergence of $\chi(y)$.

V. When $|x|$ is very large, the final integral in the equality just written is equal to $(-x)^{l'-\theta}\,\mathrm{J}(x)$, where, if $l'>l$, $|\mathrm{J}(x)|$ tends to zero as $|x|$ tends to infinity. We thus obtain a superior limit to the asymptotic value of $f_\beta(x;\theta)$ when $|x|$ is large. The problem of actually obtaining an asymptotic expansion depends upon a knowledge of the singularities of $\chi(y)$ within its circle of convergence.

Part VII.

The Functions $\sum_{n=0}^{\infty}\dfrac{x^n\Gamma(1+\alpha n)}{\Gamma(1+n)}\,(0\leqq\alpha<1)$ *and* $\sum_{n=0}^{\infty}\dfrac{x^n\Gamma(1+n\theta)}{\Gamma(1+n+n\theta)}\,(\theta>0)$.

§ 42. The asymptotic expansions of these two functions are connected with one another. The functions do not belong to the categories previously considered; their *associated* functions have not finite radius of convergence.

We give the results which may be obtained for these two functions, referring the reader elsewhere for the detailed analysis.*

I. If

$$f(x)=\sum_{n=0}^{\infty}\frac{x^n\Gamma(1+\alpha n)}{\Gamma(n+1)},\text{ we have, if }\alpha<1,\ f(x)=\int_0^{\infty}\exp\{-y+xy^{\alpha}\}\,dy,$$

the integration being taken along the positive half of the real axis.

Hence, if $\mathfrak{R}(x)<0$, we have the asymptotic expansion

$$f(x)=\frac{1}{\alpha(-x)^{1/\alpha}}\sum_{n=0}^{\infty}\frac{(-)^n\Gamma((n+1)/\alpha)}{\Gamma(n+1)(-x)^{n/\alpha}},$$

the principal value of $(-x)^{1/\alpha}$, which is real when x is real and negative, and which has a cross-cut along the positive half of the real axis, being taken.

II. When $\mathfrak{R}(x)>0$ and $|\arg x|<(1-\alpha)\frac{1}{2}\pi$, we have

$$f(x)=(\alpha x)^{\frac{1}{1-\alpha}}\int_0^{\infty}\exp\{(\alpha^{\alpha}x)^{\frac{1}{1-\alpha}}(t^{\alpha}-\alpha t)\}\,dt.$$

By the substitution $y=\dfrac{1}{\alpha}-1-\dfrac{t^{\alpha}}{\alpha}+t$ we deduce the asymptotic expansion

$$f(x)=\exp\{(1-\alpha)\,\alpha^{\frac{\alpha}{1-\alpha}}x^{\frac{1}{1-\alpha}}\}\,(\alpha x)^{\frac{1}{2(1-\alpha)}}\left(\frac{2}{1-\alpha}\right)^{1/2}\left[\Gamma(\tfrac{1}{2})+\sum_{n=1}^{\infty}\frac{c_n}{(\alpha x)^{\frac{n}{(1-\alpha)}}}\right],$$

the c_n's being definite constants.

* See a paper which will shortly appear in 'Cambridge Philosophical Transactions,' vol. 20.

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III. When $\frac{\pi}{2} \geqq |\arg x| > \frac{1}{2}(1-\alpha)\pi$, we may obtain the same expansion as in Case I.

IV. The expansions of Cases I. and III. may also be obtained by considering the contour integral

$$-\frac{1}{2\pi\iota}\int \Gamma(-s)(-x)^s \Gamma(\alpha s+1)\,ds.$$

V. When $\alpha = 1/2$, we obtain the asymptotic expansion

$$f(x) = \mathrm{P}e^{\frac{1}{4}x^2} x\Gamma(\tfrac{1}{2}) + \frac{2}{x^2}\sum_{n=0}^{\infty} \frac{(-)^n \Gamma(2n+2)}{\Gamma(n+1)\,x^{2n}},$$

valid for all values of arg x, if $\mathrm{P} = 0$ when $|\arg x| > \pi/4$.

VI. If $\phi(x)$ denote the function $\sum_{n=1}^{\infty} \frac{x^n \Gamma(1+n\theta)}{\Gamma(1+n+n\theta)}$, $\theta > 0$, we have

$$\phi(x) = \int_0^1 xy^\theta \exp\{xy^\theta(1-y)\}\,dy.$$

Hence, if $\mathfrak{R}(x) > 0$,

$$\phi(x) = \exp\{x\theta^\theta/(\theta+1)^{\theta+1}\}\sqrt{2}\{\theta^{(\theta+1)/2}x^{1/2}/(\theta+1)^{(\theta+2)/2}\} \times \left\{\Gamma(\tfrac{1}{2}) + \sum_{n=1}^{\infty}\frac{d_n}{x^n}\right\},$$

the d_n's being definite constants.

VII. We can deduce this result from the result of I. by means of the contour integral $-\frac{1}{2\pi\iota}\int \frac{\Gamma(-s)\,x^s\pi\,ds}{\Gamma(-\alpha s)\sin\pi(1-\alpha)s}$.

VIII. By considering the contour integral $\frac{1}{2\pi\iota}\int \frac{x^s\Gamma(1+\theta s)}{\Gamma(1+s+\theta s)}\frac{\pi}{\sin \pi s}ds$, we can show that, when $|\arg x| < \pi/2$,

$$\phi(-x) = -1 - \sum_{n=1}^{\infty}\frac{x^{-n}\Gamma(n+n\theta)}{\Gamma(\theta n)} - \frac{1}{\theta}\sum_{n=0}^{\infty} x^{-(n+1)/\theta}\frac{\Gamma(1+n+(n+1)/\theta)}{\Gamma(1+n)}.$$

Thus, when $\mathfrak{R}(x) < 0$, $\phi(x)$ needs two asymptotic series for its expression.

IX. The previous result can also be obtained by combining the results of III. and VII.

X. Similar analysis can be applied to series of the type

$$\sum_{n=0}^{\infty}\frac{\Gamma(1-n+q\theta n)}{\Gamma(1+\theta n)}x^n,$$

where $\theta > 0$, and q is an integer.

PART VIII.

The Function $E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1+\alpha n)}$, $\alpha > 0$.

§ 43. After the previous investigations it is natural to consider the function $\sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1+\alpha n)}$. This function has been denoted by $E_\alpha(x)$ by MITTAG-LEFFLER,* who has discussed its asymptotic behaviour. A reference to his papers will show how differently the methods of this memoir lead us to attack the problem which he has solved.

We will first consider the results which MITTAG-LEFFLER has obtained for the case $0 < \alpha < 2$. He shows that

1. $|E_\alpha(x)|$ approaches zero when $2\pi - \frac{1}{2}\alpha\pi > \phi > \frac{1}{2}\alpha\pi$, where arg $x = \phi$.
2. When $\phi = \pm\frac{1}{2}\alpha\pi$, $|E_\alpha(x)|$ approaches $\frac{1}{\alpha}$.
3. $\left|E_\alpha(x) - \frac{1}{\alpha}\exp\left[r^{\frac{1}{\alpha}}\cos\frac{\phi}{\alpha}\right]\right|$ diminishes indefinitely when $\frac{1}{2}\alpha\pi > \phi > -\frac{1}{2}\alpha\pi$.

It is evident that, where $\alpha = 0$, $E_\alpha(x) = 1/(1-x)$, and that, when $\alpha < 0$, $E_\alpha(x)$ is an asymptotic series. We assume then $\alpha > 0$.

It is evident that we may write

$$E_\alpha(-x) = \frac{1}{2\pi\iota}\int \frac{\pi x^s}{\Gamma(\alpha s+1)\sin s\pi}\,ds,$$

where the integral is taken along a contour which encloses the points $s = 0, 1, 2 \ldots \infty$, but no other poles of the subject of integration and which embraces the positive half of the real axis.

Now when $s = u + \iota v$ and $|v|$ is large, $\Gamma(s)$ behaves like $\exp\{-\frac{1}{2}\pi|v|\}$. Therefore the integral vanishes when taken along any part of an infinite contour for which $\mathfrak{R}(s) > -k$, where k is a finite positive quantity, if $|\arg x| < \frac{1}{2}\pi(2-\alpha)$. In order that this equality may have a meaning, we assume $0 < \alpha < 2$.

Hence, under these restrictions,

$$E_\alpha(-x) = \sum_{n=1}^{k} \frac{(-)^{n-1}x^{-n}}{\Gamma(1-\alpha n)} + J_k,$$

where $|J_k|$ is of order less than $\frac{1}{|x|^k}$ when $|x|$ is large.

Changing x into $-x$, we see that, if $0 < \alpha < 2$, we have the asymptotic equality

$$E_\alpha(x) = -\sum_{n=1}^{\infty} \frac{1}{x^n\Gamma(1-\alpha n)} \quad \ldots \ldots \ldots \quad (A),$$

when $2\pi - \frac{1}{2}\alpha\pi > \arg x > \frac{1}{2}\alpha\pi$.

* MITTAG-LEFFLER, 'Comptes Rendus,' tome 137, pp. 554–558, 1903.

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§ 44. *We consider next the asymptotic expansion of* $E_\alpha(x)$ *for other values of arg* x.

For this purpose we investigate the contour integral

$$\frac{1}{2\pi\iota}\int \Gamma(-\alpha s)\frac{\sin\pi(1-\alpha)s}{\sin\pi s}x^s\,ds,$$

which is taken along a contour which embraces the positive half of the real axis and encloses the poles of $\Gamma(-\alpha s)$ and the points $s = 0, 1, 2, \ldots \infty$, but no other poles of the subject of integration.

It is equal to

$$\sum_{m=0}^{\infty}\frac{(-)^{m-1}x^{\frac{m}{\alpha}}}{\alpha m\,!}\frac{\sin\pi\left(\frac{1-\alpha}{\alpha}\right)m}{\sin\pi\frac{m}{\alpha}}+\sum_{n=0}^{\infty}\frac{(-)^n\Gamma(-\alpha n)\sin\{\pi(1-\alpha)n\}x^n}{\pi}=-\frac{1}{\alpha}\exp\left[x^{\frac{1}{\alpha}}\right]+E_\alpha(x).$$

Now when $s = u+\iota v$ and $|v|$ is very large, the subject of integration tends exponentially to zero if

$$-\tfrac{1}{2}\alpha\pi+\pi|1-\alpha|-\pi+|\arg x|<0.$$

If $\alpha<1$, this condition gives $|\arg x|<\tfrac{3}{2}\pi\alpha$; and if $\alpha>1$, we must have $|\arg x|<2\pi-\tfrac{1}{2}\alpha\pi$.

If these conditions are satisfied, the integral vanishes round that part of the circle at infinity for which $u>-k$, where k is a finite positive quantity.

We may deform the contour in the usual way, and we see that the integral is equal to

$$\sum_{n=1}^{k}\frac{(-)^n\Gamma(\alpha n)\sin\pi(1-\alpha)n}{\pi x^n}+J_k,$$

where J_k denotes the integral along a line parallel to the imaginary axis cutting the real axis between $s=-k$ and $s=-(k+1)$. Thus, when $|x|$ is large, $|J_k|$ is of order less than $1/|x|^k$.

Under the assigned conditions, we therefore have the asymptotic equality

$$E_\alpha(x)-\frac{1}{\alpha}\exp\left[x^{\frac{1}{\alpha}}\right]=-\sum_{n=1}^{\infty}\frac{1}{x^n\,\Gamma(1-\alpha n)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (B),$$

in which we have at most neglected terms of order lower than any algebraical power $1/|x|$, when $|x|$ is large.

The conditions show that, when $2>\alpha\geqq 1$, the expansion is valid over the whole plane at infinity. When $1>\alpha>0$ we see that the expansion is valid over all the area at infinity not covered by the condition $2\pi-\tfrac{1}{2}\alpha\pi>\arg x>\tfrac{1}{2}\alpha\pi$ of the previous paragraph as well as over part of that area.

In the investigation of the present paragraph we may have $4>\alpha>2$. If $\alpha = 2+k$, where $k<2$, the asymptotic expansion (B) has only been proved to be valid over the area given by $|\arg x| < \pi - \frac{1}{2}k\pi$.

§ 45. It is at once evident from the expansion (B) that, if $2>\alpha>0$, we have, when $|\arg x| = \frac{1}{2}\alpha\pi$,

$$E_\alpha(x) = \frac{1}{\alpha}\exp\{\pm \iota r^{\frac{1}{\alpha}}\} - \sum_{n=1}^{k} \frac{1}{x^n \Gamma(1-\alpha n)} + J_k,$$

when $|x| = r$.

Therefore $|E_\alpha(x)|$ behaves like $\dfrac{1}{\alpha}$.

In thus finding the complete asymptotic expansions for $E_\alpha(x)$, when $0<\alpha<2$, we have incidentally verified all MITTAG-LEFFLER'S results.

§ 46. *We proceed next to consider the asymptotic value of* $E_\alpha(x)$ *when* $\alpha \geqq 2$.

In this case MITTAG-LEFFLER shows that :—

1. If $-\pi<\arg x<\pi$,

$$\left| E_\alpha(x) - \sum_{\mu} \frac{1}{\alpha} \exp\{|x|^{1/\alpha} e^{\iota\frac{2\mu\pi+\arg x}{\alpha}}\}\right|$$

diminishes indefinitely as $|x|$ increases, the summation embracing all real numbers μ, such that

$$\frac{2\mu\pi + \arg x}{\alpha} \leqq \tfrac{1}{2}\pi.$$

2. If $\arg x = \pm\pi$,

$$\left| E_\alpha(x) - \sum_{p=0}^{m-1} \frac{2}{\alpha} \exp\left\{ |x|^{1/\alpha} \cos\frac{2p+1}{\alpha}\pi \right\} \cos\left(|x|^{1/\alpha} \sin\frac{2p+1}{\alpha}\pi \right)\right|$$

(wherein $\alpha = 2m+\vartheta$, $0 \geqq \vartheta \geqq -1$ and $m = 1, 2, 3, \ldots$) diminishes indefinitely as $|x|$ increases.

§ 47. To obtain the complete asymptotic expansions which correspond to these results, we consider the contour integral

$$\frac{1}{2\pi\iota}\int \Gamma(-\alpha s)\frac{\sin\pi(q-\alpha)s}{\sin\pi s} x^s ds,$$

wherein q is an odd positive integer equal to $2p+1$ (say), and the contour of the integral embraces the positive half of the real axis, and encloses the poles of $\Gamma(-\alpha s)$ and the points $s = 0, 1, 2, \ldots \infty$, but no other poles of the subject of integration.

By CAUCHY'S Theorem of Residues the integral is equal to

$$\sum_{m=0}^{\infty} \frac{(-)^{m-1} x^{m/\alpha}}{\alpha m!} \cdot \frac{\sin\pi(qm/\alpha - m)}{\sin\pi m/\alpha} + \sum_{m=0}^{\infty} \frac{(-)^m \Gamma(-\alpha m)\sin\pi(q-\alpha) m x^m}{\pi}$$

$$= -\sum_{m=0}^{\infty} \frac{x^{m/\alpha}}{\alpha m!}\frac{\sin\pi q m/\alpha}{\sin\pi m/\alpha} + \sum_{m=0}^{\infty}\frac{x^m}{\Gamma(1+\alpha m)}.$$

Now, if $q = 2p+1$, we have

$$\frac{\sin \pi q m/\alpha}{\sin \pi m/\alpha} = e^{\pi\iota(q-1)m/\alpha} \cdot \frac{1-e^{-2\pi\iota q m/\alpha}}{1-e^{-2\pi\iota m/\alpha}}$$
$$= \sum_{\mu=-p}^{p} e^{2\pi\iota\mu m/\alpha}.$$

Hence the integral is equal to

$$\mathrm{E}_\alpha(x) - (1/\alpha) \sum_{\mu=-p}^{p} \exp\{x^{1/\alpha} e^{2\pi\iota\mu/\alpha}\}.$$

Now, if $s = u + \iota v$, the integral will tend exponentially to zero when $|v|$ is large if

$$-\tfrac{1}{2}\alpha\pi + |2p+1-\alpha|\pi - \pi + |\arg x| < 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

We suppose in the integral that $x^s = \exp\{s \log x\}$, and we take that value of $\arg x$ which is such that $|\arg x| \lesseqgtr \pi$.

If now $\alpha > 2p+1$, we obtain from the condition (1)

$$\tfrac{1}{2}\alpha\pi - 2(p+1)\pi + |\arg x| < 0,$$

and this will hold for all values of $|\arg x|$ which are $\lesseqgtr \pi$, provided $\tfrac{1}{2}\alpha < 2p+1$.

On the other hand, if $\alpha < 2p+1$, we must have

$$2p\pi - \tfrac{3}{2}\pi\alpha + |\arg x| < 0,$$

and this will hold for all values of $|\arg x|$ which are $\lesseqgtr \pi$ if $2p+1 < \tfrac{3}{2}\alpha$.

The contour integral will therefore vanish, when taken around that part of the circle at infinity bounded by $s = -k$, when k is a finite positive quantity, for all values of $|\arg x|$ which are $\lesseqgtr \pi$ if we take p such that either

$$\tfrac{1}{2}\alpha < 2p+1 < \alpha,$$

or

$$\alpha < 2p+1 < \tfrac{3}{2}\alpha.$$

In either case therefore the integral will be equal to

$$-\sum_{m=1}^{k} \frac{\Gamma(\alpha m)(-)^m \sin \pi(\alpha - q)m}{\pi x^m} + \mathrm{J}_k,$$

where, when $|x|$ is large, $|\mathrm{J}_k|$ is of lower order than $1/|x|^k$.

We therefore have

$$\mathrm{E}_\alpha(x) - \sum_{\mu=-p}^{p} \frac{1}{\alpha} \exp\{x^{1/\alpha} e^{2\pi\iota\mu/\alpha}\} = -\sum_{m=1}^{k} \frac{1}{\Gamma(1-\alpha m)\, x^m} + \mathrm{J}_k \quad . \quad . \quad . \quad (\mathrm{C}).$$

This asymptotic equality is valid for all values of $\arg x$ between $\pm\pi$ (the limits included) if p be so chosen that either

$$\frac{\alpha}{2} < 2p+1 < \alpha,$$

or

$$\alpha < 2p+1 < \tfrac{3}{2}\alpha.$$

Apparently, then, the expansion is not unique. But this indeterminateness is illusory, for it only corresponds to terms of the sum

$$\sum_{\mu=-p}^{p} \exp\{x^{1/\alpha}e^{2\pi\iota\mu/\alpha}\},$$

for which $\{2\pi\mu + \arg x\}/\alpha$ does not lie between $\pm\frac{1}{2}\pi$.

If we neglect these terms, which may be absorbed in J_k, we may say that $E_\alpha(x)$ admits the asymptotic expansion

$$\frac{1}{\alpha}\sum_{\mu} \exp\{x^{1/\alpha}e^{2\pi\iota\mu/\alpha}\} - \sum_{m=1}^{\infty}\frac{1}{\Gamma(1-\alpha m)\, x^m},$$

wherein μ takes all integral values (positive, negative, or zero) such that

$$2\pi\mu + \arg x \lesseqgtr \tfrac{1}{2}\alpha\pi,$$

arg x having any value between $\pm\pi$, these limits included.

This is equivalent to MITTAG-LEFFLER'S results.

The Function $E_\alpha(x;\theta,\beta) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(1+\alpha n)(\theta+n)^\beta}$.

§ 48. It is evident that the results just obtained admit of many extensions. As typical of these we may consider the function

$$E_\alpha(x;\theta,\beta) = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(1+\alpha n)(\theta+n)^\beta},$$

wherein $\alpha > 0$ and θ is not equal to zero or a negative integer.

We see at once that

$$E_\alpha(-x;\theta,\beta) = \frac{1}{2\pi\iota}\int\frac{\pi x^s}{\Gamma(\alpha s+1)\sin\pi s\,(s+\theta)^\beta}\,ds,$$

the integral being taken round a contour which embraces the positive half of the real axis and includes the points $s = 0, 1, 2, \ldots \infty$, but no other poles of the subject of integration.

We assume that $x^s = \exp\{s\log x\}$, and that the logarithm has its principal value whose argument lies between $\pm\pi$. We also assume that $(s+\theta)^\beta = \exp\{\beta\log(s+\theta)\}$, and that if θ be not real, the logarithm has its principal value with respect to a cross-cut drawn from $s = -\theta$, parallel to the negative direction of the real axis. If θ were real, we should slightly deform this cross-cut. We omit the consideration of this particular case in the following investigation.

We assume in the first place that $2 > \alpha > 0$. Then if $|\arg x| < \pi(1-\frac{1}{2}\alpha)$, the integral vanishes along that part of a great circle at infinity for which $\mathfrak{R}(s) > -k$.

We therefore have

$$E_{\alpha}(-x\,;\,\theta,\beta) = -\sum_{n=1}^{\infty}\frac{(-)^n x^{-n}}{\Gamma(1-\alpha n)(\theta-n)^{\beta}} - \frac{1}{2\pi\iota}\int\frac{x^{-y-\theta}(-y)^{-\beta}\pi\,dy}{\Gamma(1-\alpha\theta-\alpha y)\sin\pi(\theta+y)},$$

the latter integral embracing the positive half of the real axis and including the origin, but no other singularity of the subject of integration. Since the zeros of $\Gamma(1-\alpha\theta-\alpha y)\sin\pi(\theta+y)$ lie outside the contour, we may employ the summable divergent series

$$\pi/\{\Gamma(1-\alpha\theta-\alpha y)\sin\pi(\theta+y)\} = \sum_{n=0}^{\infty}(-)^n d_n y^n$$

under the sign of integration. The integral will then be represented asymptotically by

$$-\frac{x^{-\theta}}{2\pi\iota}\sum_{n=0}^{\infty}\int(-y)^{-\beta+n}d_n e^{-y\log x}\,dy = -x^{-\theta}\sum_{n=0}^{\infty}\frac{d_n}{\Gamma(\beta-n)(\log x)^{n-\beta+1}}.$$

If then $|\arg x| < \pi(1-\alpha/2)$, we have the asymptotic expansion

$$E_{\alpha}(-x\,;\,\theta,\beta) = -\sum_{n=1}^{\infty}\frac{1}{\Gamma(1-\alpha n)(\theta-n)^{\beta}(-x)^n} + x^{-\theta}(\log x)^{\beta-1}\sum_{n=0}^{\infty}\frac{d_n}{\Gamma(\beta-n)(\log x)^n}.$$

Therefore when $0<\alpha<2$, and $2\pi-\frac{1}{2}\alpha\pi > \arg x > \frac{1}{2}\alpha\pi$, we have

$$E_{\alpha}(x;\theta,\beta) = -\sum_{n=1}^{\infty}\frac{1}{x^n\Gamma(1-\alpha n)(\theta-n)^{\beta}} + (-x)^{-\theta}\{\log(-x)\}^{\beta-1}\sum_{n=0}^{\infty}\frac{d_n}{\Gamma(\beta-n)[\log(-x)]^n} \quad \text{(A).}$$

In this formula the principal value of $\log(-x)$, whose imaginary part lies between $\pm\pi$, is to be taken, and $(\theta-n)^{\beta}$ is defined with reference to the cross-cut previously taken.

§ 49. To obtain the asymptotic expansion of $E_{\alpha}(x\,;\,\theta,\beta)$ for other values of $|\arg x|$ when $0<\alpha<2$, and for all values of $\arg x$ when $\alpha>2$, we consider the contour integral

$$\frac{1}{2\pi\iota}\int\Gamma(-\alpha s)\frac{\sin\{\pi(q-\alpha)s\}\,x^s}{\sin\pi s\,(\theta+s)^{\beta}}\,ds$$

round a contour which embraces the positive half of the real axis and includes the poles of $\Gamma(-\alpha s)$ and the points $s = 0, 1, 2, \ldots \infty$, but no other singularities of the subject of integration.

In the subject of integration q is an odd positive integer equal to $2p+1$ (say), and x^s and $(\theta+s)^{\beta}$ are defined as in the previous section. The integral is evidently equal to

$$-\frac{1}{\alpha}\sum_{m=0}^{\infty}\frac{x^{m/\alpha}}{m!}\frac{\sin\pi q m/\alpha}{\sin\pi m/\alpha\,(\theta+m/\alpha)^{\beta}} + \sum_{m=0}^{\infty}\frac{x^m}{\Gamma(1+\alpha m)(\theta+m)^{\beta}}.$$

The first series is equal to

$$-\frac{1}{\alpha}\sum_{\mu=-p}^{p}\sum_{m=0}^{\infty}\frac{x^{m/\alpha}e^{2\pi\iota m\mu/\alpha}}{m!\,(\theta+m/\alpha)^{\beta}} = -\alpha^{\beta-1}\sum_{\mu=-p}^{p}G_{\beta}\{x^{1/\alpha}e^{2\pi\iota\mu/\alpha}\,;\,\alpha\theta\},$$

where

$$G_\beta(x;\theta) = \sum_{m=0}^{\infty} \frac{x^m}{m!\,(m+\theta)^\beta}.$$

Now we have obtained the asymptotic expansion (§ 22)

$$G_\beta(x;\theta) = \frac{e^x}{x^\beta}\sum_{n=0}^{\infty}\frac{c_n\Gamma(1-\beta)}{x^n\Gamma(1-\beta-n)} + (-x)^{-\theta}[\log(-x)]^{\beta-1}\sum_{n=0}^{\infty}\frac{(-)^n\Gamma^{(n)}(\theta)}{n!\,\Gamma(\beta-n)[\log(-x)]^n},$$

where the c's are defined by the expansion, valid when $y<1$,

$$\left[\frac{\log(1-y)}{-y}\right]^{\beta-1}(1-y)^{\theta-1} = \sum_{n=0}^{\infty} c_n(-y)^n.$$

If $|\arg x| \leqq \pi$, we see, as before, that it is sufficient to take p such that $\frac{1}{2}\alpha < 2p+1 < \alpha$, or $\alpha < 2p+1 < \frac{3}{2}\alpha$, in order that the integral may vanish for an infinite contour for which $\mathfrak{R}(s) > -k$.

When $1<\alpha<2$, we may take $p=0$. And when $0<\alpha \leqq 1$, we may take $p=0$, provided $|\arg x| < \frac{3}{2}\alpha\pi$.

Under these conditions the integral is asymptotically equal to

$$-\sum_{m=1}^{\infty}\frac{1}{\Gamma(1-\alpha m)(\theta-m)^\beta x^m} - \frac{x^{-\theta}}{2\pi\iota}\int x^{-y}(-y)^{-\beta}\frac{\Gamma(\alpha\theta+\alpha y)\sin\{\pi(\alpha-q)(y+\theta)\}}{\sin\pi(\theta+y)}\,dy.$$

The contour of the integral embraces the positive half of the real axis and contains no singularities of the subject of integration except the origin. The terms neglected in the equality are of order less than any algebraical power of $1/|x|$, when $|x|$ is large.

To obtain an asymptotic expansion for the integral, we may, under the integral sign, employ the summable divergent expansion

$$\frac{\Gamma(\alpha\theta+\alpha y)\sin\{\pi(\alpha-q)(y+\theta)\}}{\sin\pi(\theta+y)} = \sum_{n=0}^{\infty} e_n(-y)^n.$$

The integral is then represented by $x^{-\theta}\sum_{n=0}^{\infty}\frac{e_n}{\Gamma(\beta-n)[\log x]^{n-\beta+1}}$.

All these asymptotic expansions are negligible compared with the dominant terms of $G_\beta\{x^{1/\alpha}e^{2\pi\iota\mu/\alpha};\alpha\theta\}$.

Hence when $0<\alpha<2$, and $|\arg x| < \alpha\pi/2$, we have asymptotically

$$E_\alpha(x;\theta,\beta) = \alpha^{\beta-1}\frac{\exp\{x^{1/\alpha}\}}{x^{\beta/\alpha}}\sum_{n=0}^{\infty}\frac{{}_\theta c_n\Gamma(1-\beta)}{x^{n/\alpha}\Gamma(1-\beta-n)},$$

where ${}_\theta c_n$ is given by the expansion

$$[\log(1-y)]^{\beta-1}(1-y)^{\alpha\theta-1} = (-y)^{\beta-1}\sum_{n=0}^{\infty}{}_\theta c_n(-y)^n,$$

valid when $|y|<1$.

And when $\alpha \gtrsim 2$, we have asymptotically for all values of $|\arg x|$

$$\mathrm{E}_{\alpha}(x\,;\theta,\beta] = \alpha^{\beta-1} \sum_{\mu=-p}^{p} \left[\frac{\exp\{x^{1/\alpha} e^{\frac{2\pi\iota\mu}{\alpha}}\}}{x^{\beta/\alpha} e^{2\pi\iota\mu\beta/\alpha}} \sum_{n=0}^{\infty} \frac{{}_{\theta}c_n \Gamma(1-\beta)}{x^{n/\alpha} e^{2\pi\iota\mu n/\alpha} \Gamma(1-\beta-n)}\right],$$

wherein μ takes all integral values, positive, negative, or zero, such that

$$2\pi\mu + \arg x \lesssim \tfrac{1}{2}\alpha\pi.$$

PART IX.

The Function ${}_1\mathrm{F}_1\{\alpha\,;\rho\,;x\}$.

§ 50. Generalised hypergeometric functions form a wide class of integral functions whose asymptotic expansions are closely connected with the theory of linear differential equations.

The general type of such series is

$$1 + \frac{\alpha_1 \ldots \alpha_p}{1\,.\,\rho_1 \ldots \rho_q} x + \frac{\alpha_1(\alpha_1+1)\ldots\alpha_p(\alpha_p+1)}{1\,.\,2\,.\,\rho_1(\rho_1+1)\ldots\rho_q(\rho_q+1)} x^2 + \ldots$$

$$= \frac{\Gamma(\rho_1)\ldots\Gamma(\rho_q)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n)\ldots\Gamma(\alpha_p+n)}{\Gamma(n+1)\,\Gamma(\rho_1+n)\ldots\Gamma(\rho_q+n)} x^n,$$

wherein $p \lesssim q$.

This series we shall denote by ${}_p\mathrm{F}_q\{\alpha_1, \ldots, \alpha_p\,;\rho_1, \ldots, \rho_q\,;x\}$ or briefly by ${}_p\mathrm{F}_q\{x\}$.

The series satisfies the differential equation

$$\left\{(\vartheta+\alpha_1)\ldots(\vartheta+\alpha_p) - \frac{d}{dx}(\vartheta+\rho_1-1)\ldots(\vartheta+\rho_q-1)\right\} y = 0 \quad . \quad . \quad . \quad (1),$$

where $\vartheta = x\dfrac{d}{dx}$. The equation is of order $(q+1)$, and the other q solutions are given by

$$x^{1-\rho_1}{}_p\mathrm{F}_q\{\alpha_1-\rho_1+1, \ldots, \alpha_p-\rho_1+1\,; 2-\rho_1, \rho_2-\rho_1+1, \ldots, \rho_q-\rho_1+1\,; x\},$$

and $(q-1)$ similar functions.

I give here some of the results which I have obtained by applying integrals of the types previously used to the theory of these series. For detailed investigations I refer to a forthcoming paper.*

I. The series ${}_1\mathrm{F}_1\{\alpha\,;\rho\,;x\}$ satisfies the equation

$$x\frac{d^2y}{dx^2} - (x-\rho)\frac{dy}{dx} - \alpha y = 0.$$

* 'Cambridge Philosophical Transactions,' vol. 20.

An independent solution of this equation is

$$x^{1-\rho}{}_1F_1(\alpha-\rho+1\,;\,2-\rho\,;\,x).$$

II. By considering the contour integral $-\dfrac{1}{2\pi\iota}\displaystyle\int\Gamma(n-s)\,x^s\,ds$, we may show that

$${}_1F_1\{\alpha\,;\,\rho\,;\,x\} = e^x{}_1F_1\{\rho-\alpha\,;\,\rho\,;\,-x\}.$$

This result is valid for all values of $\arg x$. It has been otherwise obtained by Orr.

III. By considering the contour integral

$$-\frac{1}{2\pi\iota}\int\frac{\Gamma(-s)\,\Gamma(\alpha+s)\,(-x)^s}{\Gamma(\rho+s)}\,ds,$$

we may show that, when $\mathfrak{R}(x) < 0$, ${}_1F_1\{\alpha\,;\,\rho\,;\,x\}$ admits the asymptotic expansion

$$[\{\Gamma(\rho)/\Gamma(\rho-\alpha)\}]\,(-x)^{-\alpha}{}_2F_0\left\{\alpha,\ 1-\rho+\alpha\,;\,-\frac{1}{x}\right\}.$$

IV. Combining II. and III., we show that, when $\mathfrak{R}(x) > 0$, ${}_1F_1\{\alpha\,;\,\rho\,;\,x\}$ admits the asymptotic expansion

$$\frac{e^x\Gamma(\rho)}{\Gamma(\alpha)\,x^{\rho-\alpha}}\,{}_2F_0\left\{\rho-\alpha,\ 1-\alpha\,;\,\frac{1}{x}\right\}.$$

V. The combination of III. and IV. gives us the complete asymptotic expansion of ${}_1F_1\{\alpha\,;\,\rho\,;\,x\}$.

This I have verified by means of integrals taken round double loop contours of Pochhammer's type.

VI. It is possible to take such a linear combination of the two solutions

$${}_1F_1\{\alpha\,;\,\rho\,;\,x\} \quad\text{and}\quad {}_1F_1\{\alpha-\rho+1\,;\,2-\rho\,;\,x\}\,x^{1-\rho}$$

as will admit all round $x = \infty$ the single asymptotic expansion

$$(-x)^{-\alpha}{}_2F_0\{\alpha,\ 1-\rho+\alpha\,;\,-1/x\},$$

By considering the contour integral

$$\frac{1}{2\pi\iota}\int\Gamma(-s)\,\Gamma(1-\rho-s)\,\Gamma(\alpha+s)\,x^s\,ds$$

we can, in fact, prove that, when $|\arg x| < 3\pi/2$,

$$\Gamma(\alpha)\,\Gamma(1-\rho)\,{}_1F_1\{\alpha\,;\,\rho\,;\,x\}+\Gamma(\alpha+1-\rho)\,\Gamma(\rho-1)\,x^{1-\rho}{}_1F_1\{\alpha-\rho+1\,;\,2-\rho\,;\,x\}$$
$$= x^{-\alpha}\Gamma(\alpha)\,\Gamma(1+\alpha-\rho)\,{}_2F_0\{\alpha,\ 1+\alpha-\rho\,;\,-1/x\}.$$

This theorem is equivalent to two different results when $\mathfrak{R}(x) < 0$. By this means we can obtain III. anew.

VII. Similarly, when $|\arg x| < 3\pi/2$, we have

$$\frac{\Gamma(1-\rho)}{\Gamma(1-\alpha)}\,{}_1F_1\{\alpha\,;\rho\,;-x\} + \frac{\Gamma(\rho-1)}{\Gamma(\rho-\alpha)}\,x^{1-\rho}{}_1F_1\{1-\rho+\alpha\,;2-\rho\,;-x\}$$
$$= e^{-x}x^{\alpha-\rho}{}_2F_0\left\{\rho-\alpha,\ 1-\alpha\,;-\frac{1}{x}\right\}.$$

VIII. If we put $\alpha = 1$, we obtain the function $F_\rho(x) = \sum_{n=0}^{\infty} \frac{x^n\Gamma(\rho)}{\Gamma(\rho+n)}$. For this function we obtain, when $|\arg x| < \pi$, the asymptotic equality

$$F_\rho(x) = e^x x^{1-\rho}\Gamma(\rho) - \sum_{n=1}^{\infty} \frac{(\rho-1)\ldots(\rho-n)}{x^n}.$$

This result can be otherwise obtained from the equality

$$F_\rho(x) = 1 + xe^x\int_0^1 e^{-xy}y^{\rho-1}\,dy.$$

PART X.

The Function ${}_0F_1\{\rho\,;x\}$.

§ 51. I. The function ${}_0F_1\{\rho\,;x\} = \Gamma(\rho)\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n+1)\Gamma(\rho+n)}$, wherein ρ may have any value which is not zero or a negative integer, satisfies the differential equation

$$x\frac{d^2y}{dx^2} + \rho\frac{dy}{dx} - y = 0.$$

A second independent solution is $x^{1-\rho}{}_0F_1\{2-\rho\,;x\}$. Evidently the function is substantially BESSEL's function.

We have

$$J_n(z) = \frac{1}{\Gamma(n+1)}\left(\tfrac{1}{2}z\right)^n{}_0F_1\{n+1\,;-\tfrac{1}{4}z^2\}.$$

II. By considering the contour integral $-\frac{1}{2\pi\iota}\int \Gamma(2n-s)\,(4x)\,\tfrac{1}{2}s\,ds$, we prove that

$${}_0F_1\{\rho\,;x\} = e^{-2x^{1/2}}{}_1F_1\{\rho-\tfrac{1}{2}\,;\ 2\rho-1\,;\ 4x^{1/2}\}$$
$$= e^{2x^{1/2}}{}_1F_1\{\rho-\tfrac{1}{2}\,;\ 2\rho-1\,;\ -4x^{1/2}\}.$$

This result is valid for all values of arg x, and for the case of real variables was first established by KUMMER.

By means of this theorem we deduce the theory of the function ${}_0F_1\{\rho\,;x\}$ from the theory of the function ${}_1F_1\{\alpha\,;\rho\,;x\}$ developed in Part IX.

III. For all values of arg x we have the asymptotic expansion

$$ {}_0F_1\{\rho\,;\,x\} = e^{2x^{1/2}}\frac{\Gamma(2\rho-1)}{\Gamma(\rho-\frac{1}{2})}(4x^{1/2})^{1/2-\rho}{}_2F_0\left\{\rho-\tfrac{1}{2},\ \tfrac{3}{2}-\rho\,;\ \frac{1}{4x^{1/2}}\right\} $$
$$ +\, e^{-2x^{1/2}}\frac{\Gamma(2\rho-1)}{\Gamma(\rho-\frac{1}{2})}(-4x^{1/2})^{1/2-\rho}{}_2F_0\left\{\rho-\tfrac{1}{2},\ \tfrac{3}{2}-\rho\,;\ \frac{1}{4x^{1/2}}\right\}. $$

We take that value of $(4x^{1/2})^{1/2-\rho}$ which is equal to $\exp\{(\frac{1}{2}-\rho)\log(4x^{1/2})\}$, the logarithm being real when x is real and positive and having a cross-cut along the negative half of the real axis. Similarly $(-4x^{1/2})^{1/2-\rho}$ is equal to $\exp\{(\frac{1}{2}-\rho)\log(-4x^{1/2})\}$ when the logarithm is real, when x is real and negative and has a cross-cut along the positive half of the real axis.

IV. We may deduce the asymptotic behaviour of BESSEL'S function.

The theorem, though its expression is more concise, is equivalent to the results obtained by STOKES.

V. By considering the integral $-\frac{1}{2\pi\iota}\int\Gamma(-s)\,\Gamma(1-\rho-s)\,x^s\,ds$ we may prove that, if $|\arg x|<\pi$,

$$ \Gamma(1-\rho)\,{}_0F_1\{\rho\,;\,x\}+x^{1-\rho}\,\Gamma(\rho-1)\,{}_0F_1\{2-\rho\,;\,x\} $$
$$ = \sqrt{\pi}x^{(1-2\rho)/4}e^{-2x^{1/2}}{}_2F_0\{\tfrac{3}{2}-\rho\,;\ \rho-\tfrac{1}{2}\,;\ -\tfrac{1}{4}x^{-1/2}\}. $$

PART XI.

The Generalised Hypergeometric Functions.

§ 52. When $p \leqq q$, the generalised hypergeometric functions are integral functions. We limit ourselves to this case.

I. By considering the contour integral

$$ -\frac{1}{2\pi\iota}\int\frac{\Gamma(-s)\,\Gamma(\alpha_1+s)\,\Gamma(1-\rho_1-s)\ldots\Gamma(1-\rho_q-s)}{\Gamma(1-\alpha_2-s)\ldots\Gamma(1-\alpha_p-s)}x^s\,ds $$

we may show that the linear combination of functions,

$$ \frac{\Gamma(\alpha_1)\,\Gamma(1-\rho_1)\ldots\Gamma(1-\rho_q)}{\Gamma(1-\alpha_2)\ldots\Gamma(1-\alpha_p)}\,{}_pF_q\{\alpha_1,\ldots,\ \alpha_p\,;\ \rho_1,\ldots,\ \rho_q\,;\ (-)^{p-q}x\} $$
$$ +\,\frac{\Gamma(\alpha_1-\rho_1+1)\,\Gamma(\rho_1-1)\,\Gamma(\rho_1-\rho_2)\ldots\Gamma(\rho_1-\rho_q)}{\Gamma(\rho_1-\alpha_2)\,\Gamma(\rho_1-\alpha_3)\ldots\Gamma(\rho_1-\alpha_p)}x^{1-\rho_1} $$
$$ {}_pF_q\{\alpha_1-\rho_1+1,\ldots,\ \alpha_p-\rho_1+1\,;\ 2-\rho_1,\ 1-\rho_1+\rho_2,\ldots,\ 1-\rho_1+\rho_q\,;\ (-)^{p-q}x\} $$

+ $(q-1)$ other terms similar to the last, admits the asymptotic expansion

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_1-\rho_1+1)\ldots\Gamma(\alpha_1-\rho_q+1)}{\Gamma(1-\alpha_2+\alpha_1)\ldots\Gamma(1-\alpha_p+\alpha_1)}x^{-\alpha_1}$$

$$\times\,{}_{q+1}F_{p-1}\left\{\alpha_1,\ \alpha_1-\rho_1+1,\ldots,\ \alpha_1-\rho_q+1\,;\ \alpha_1-\alpha_2+1,\ldots,\ \alpha_1-\alpha_p+1\,;\ -\frac{1}{x}\right\},$$

provided $|\arg x| < \frac{1}{2}(q-p+3)\pi$.

There are evidently p relations similar to the one just written, each corresponding to an asymptotic solution of the differential equation (1) of p in the neighbourhood of $x=\infty$. There are therefore $q+1-p$ other asymptotic solutions near $x=\infty$ which will be asymptotic expansions of other linear combinations of the $q+1$ fundamental hypergeometric functions.

[II.] The linear combination of hypergeometric functions

$$\frac{\prod_{r=1}^{q}\Gamma(1-\rho_r)}{\prod_{r=1}^{p}\Gamma(1-\alpha_r)}\,{}_pF_q\{\alpha_1,\ldots,\ \alpha_p\,;\ \rho_1,\ldots,\ \rho_q\,;\ (-)^{-p+1+q}x\}$$

$$+\sum_{r=1}^{q}x^{1-\rho_r}\frac{\Gamma(\rho_r-1)\prod_{t=1}^{q}{}'\,\Gamma(\rho_r-\rho_t)}{\prod_{t=1}^{p}\Gamma(\rho_r-\alpha_t)}\,{}_pF_q\{1-\alpha_1-\rho_r,\ldots,\ 1+\alpha_p-\rho_r\,;$$

$$2-\rho_r,\ldots,\ \rho_q-\rho_r+1\,;\ (-)^{q+1-p}x\}$$

can be expressed by the contour integral

$$I=-\frac{1}{2\pi\iota}\int\frac{\Gamma(-s)\prod_{r=1}^{q}\Gamma(1-\rho_r-s)}{\prod_{r=1}^{p}\Gamma(1-\alpha_r-s)}x^s\,ds,$$

and provided $|\arg x| < (q+1-p)\pi/2$, this integral may be taken along a line in the finite part of the plane parallel to the imaginary axis.

If $q+1-p=\mu$, we obtain

$$\exp\{\mu x^{1/\mu}\}(2\pi)^{(1-\mu)/2}\mu^{1/2}\,I=-\frac{1}{2\pi\iota}\int S(s)\,x^s\,ds,$$

where

$$S(s)=\sum_{t=0}^{\infty}\frac{\Gamma(-s+t/\mu)\prod_{r=1}^{q}\Gamma(1-\rho_r+t/\mu-s)}{\prod_{r=1}^{\mu}\Gamma\left(\frac{t+r}{\mu}\right)\prod_{r=1}^{p}\Gamma\left(1-\alpha_r+\frac{t}{\mu}-s\right)},$$

when

$$\mathfrak{R}(s)>\mathfrak{R}\left\{\frac{\mu-1}{2}+\alpha_1+\ldots+\alpha_p-\rho_1-\ldots-\rho_q\right\}/\mu.$$

We can infer that asymptotically, when $|x|$ is large,

$$I = \exp\{-\mu x^{1/\mu}\}(2\pi)^{(\mu-1)/2}\mu^{-1/2}x^{\{\Sigma\alpha-\Sigma\rho+(\mu-1)/2\}/\mu}J(x),$$

where $|J(x)|$ tends to unity as $|x|$ tends to infinity.

The complete asymptotic expansion is thus made to depend upon the determination of the singularities of S (s).

The relation obtained holds when $|x^{1/\mu}| < \pi$, and is thus equivalent to $q+1-p$ independent relations.

In this way the $(q+1)$ asymptotic expansions of the differential equation for the generalised hypergeometric equation are connected with the solutions which are integral functions in the finite part of the plane. The results agree with those of ORR; the methods, however, which have been suggested enable us to dispense entirely with his elaborate analysis.

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